INTEGRATION THEORY AND FUNCTIONAL ANALYSIS

M.A./M.Sc. Mathematics (Final)

MM-501

Directorate of Distance Education
Maharshi Dayanand University
ROHTAK – 124 001

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MM-501

Max. Marks : 100

Time : 3 Hours

Note: Question paper will consist of three sections. Section I consisting of one question with ten parts covering whole of the syllabus of 2 marks each shall be compulsory. From Section II, 10 questions to be set selecting two questions from each unit. The candidate will be required to attempt any seven questions each of five marks. Section III, five questions to be set, one from each unit. The candidate will be required to attempt any three questions each of fifteen marks.

Unit I

Signed measure, Hahn decomposition theorem, Jordan decomposition theorem, Mutually singular measure, Radon-Nikodym theorem. Lebesgue decomposition, Lebesgue-Stieltjes integral, Product measures, Fubini's theorem.

Baire sets, Baire measure, Continuous functions with compact support, Regularity of measures on locally compact support, Riesz-Markoff theorem.

Unit II

Normed linear spaces, Metric on normed linear spaces, Holder's and Minkowski's inequality, Completeness of quotient spaces of normed linear spaces. Completeness of l_p, L^p, R^n, C^n and C [a, b]. Bounded linear transformation. Equivalent formulation of continuity. Spaces of bounded linear transformations, Continuous linear functional, Conjugate spaces, Hahn-Banach extension theorem (Real and Complex form), Riesz Representation theorem for bounded linear functionals on L^p and C[a,b].

Unit III

Second conjugate spaces, Reflexive spaces, Uniform boundedness principle and its consequences, Open mapping theorem and its application, projections, Closed Graph theorem, Equivalent norms, weak and strong convergence, their equivalence in finite dimensional spaces.

Unit IV

Compact operations and its relation with continuous operator. Compactness of linear transformation on a finite dimensional space, properties of compact operators, Compactness of the limit of the sequence of compact operators. The closed range theorem.

Inner product spaces, Hilbert spaces, Schwarz's inequality, Hilbert space as normed linear space, Convex sets in Hilbert spaces. Projection theorem.

Unit V

Orthonormal sets, Bessell's inequality, Parseval's identity, Conjugate of Hilbert space, Riesz representation theorem in Hilbert spaces. Adjoint of an opertor on a Hilbert space, Reflexivity of Hilbert space, Self-adjoint operator, Positive operator, Normal and unitary operators, Projections on Hilbert space, Spectral theorem on finite dimensional spaces, Lax-Milgiam theorem.

Unit-I

Signed Measure

Signed Measure

We define a measure as a non-negative set function, we will now allow measure to take both positive and negative values.

Suppose that μ_1 and μ_2 are two measures, on the same measurable space (X, B). If we define a new measure μ_3 on (X, B) by setting

$$\mu_3(E) = C_1\, \mu_1(E) + C_2\, \mu_2(E)\ C_1, C_2 \geq 0.$$

Then it is clear that μ_3 is a measure, thus two measures can be added. This can be extended to any finite sum.

Another way of constructing new measures is to multiply a given measure by an arbitrary non-negative constant. Combining these two methods, we see that if

$$\{\mu_1, \mu_2, \ldots, \mu_n\}.$$

is a finite set of measures and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite set of non negative real numbers. Then the set fn μ defined for every set E in X by

$$\mu\, E = \sum_{i=1}^{n} \alpha_i\, \mu_i\, E$$

is a measure.

Now what happens if we try to define a measure by

$$vE = \mu_1 E - \mu_2 E$$

The first thing may occur is that v is not always non-negative and this leads to the consideration of signed measure which we shall define now. Also we get more difficulty from the fact that v is not defined when $\mu_1 E = \mu_2 E = \infty$. For this reason, we should have either $\mu_1 E$ or $\mu_2 E$ finite with these consideration in mind, we make the following definition

Definition :- Let (X, B) be a measurable space. An extended real valued function, $v : B \rightarrow R$ defined on the σ–algebra B is called a signed measure if it satisfies the following conditions.

(1) v assumes at most one of the values $+\infty$ and $-\infty$

(2) $v(\phi) = 0$

(3) For any sequence $\{E_i\}$ of disjoint measurable sets.

$$v\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} vE_i$$

the equality here means that the series on the R.H.S. converges absolutely if $v\left(\bigcup_{i=1}^{\infty} E_i\right)$ is finite and that it properly diverges otherwise i.e. definitely diverges to $+\infty$ or $-\infty$.

Thus a measure is a special case of signed measure but a signed measure is not in general a measure.

Definition :- Let (X, B) be a measurable space and let A be a subspace of X. We say that A is a positive set w.r. to signed measure v if A is measurable and for every measurable subset E of A we have $vE \geq 0$. Every measurable subset of positive set is again positive and if we take the restriction of v to a positive set, we obtain a measure. Similarly a set B is called negative if it is measurable and every measurable subset E of it has a non-positive v measure i.e. $vE \leq 0$.

A set which is both positive and negative with respect to v is called a null set. Thus a measurable set is a null set iff every measurable subset of it has v measure zero.

Remark :- Every null set have measure zero but a set of measure zero may be a union of two sets whose measures are not zero but are negatives of each other. Similarly a positive set is not to be confused with a set which merely has positive measure.

Lemma 1 :- The union of a countable collection of positive sets is positive.

Proof :- Let $A = \cup A_n$ be the union of a sequence $\langle A_n \rangle$ of positive sets. Let E be a measurable subset of A. Since A_n are measurable, A is measurable and A_n^c are measurable. Set

$$E_n = E \cap A_n^C \cap A_{n-1}^C \cap \ldots \cap A_1^C$$

Then E_n is a measurable subset of A_n and $vE_n \geq 0$. Since the sets E_n's are disjoint and. $E = \cup E_n$.

Therefore we have

$$v(E) = \sum_{n=1}^{\infty} v(E_n) \geq 0.$$

Thus we have proved that $A = \cup A_n$ is a measurable set and for every measurable subset E of A we have $v(E) \geq 0$. Hence A is a positive set.

Lemma 2 :- If E and F are measurable sets and v is a signed measure such that $E \subset F$ and $|vF| < \infty$

Then $|v E| < \infty$.

Proof :- We have $vF = v(F-E) + v(E)$

If exactly one of the term is infinite then so is v(F). If they are both infinite, [since v assumes at most one of the values $+\infty$ and $-\infty$.) They are equal and again infinite.

Thus only one possibility remains that both terms are finite and this proves that every measurable subset of a set of finite signed measure has finite signed measure.

Theorem 1 :- Let E be a measurable set such that $0 < vE < \infty$. Then there is a positive set A contained in E with $vA > 0$.

Proof :- If E is a positive set then we take A = E and thus $vA = vE > 0$ which proves the theorem.

We consider the case when E is not positive, then it contains sets of negative measure. Let n_1 be the smallest positive integer such that there is a measurable set $E_1 \subset E$ with $vE_1 < -\dfrac{1}{n_1}$

Now $E = (E-E_1) \cup E_1$

and $E-E_1$ and E_1 are disjoint

Therefore $vE = v(E-E_1) + v(E_1)$

$\Rightarrow \quad v(E-E_1) = vE - vE_1 \quad \ldots(1)$

Since vE is finite (given). It follows that $v(E-E_1)$ and vE_1 are finite.

Moreover $vE > 0$ and vE_1 is negative, it follows from (1) that $v(E-E_1) > 0$.

Thus $0 < v(E-E_1) < \infty$.

If $E-E_1$ is positive, we can take

$A = E - E_1$. Hence the result

Suppose that $E - E_1$ is not positive. Then it contains set of negative measure. Let n_2 be the smallest positive integer such that there is a measurable set $E_2 \subset E - E_1$ with

$$vE_2 < -\frac{1}{n_2}$$

Now $$E = \left[E - \left(\bigcup_{i=1}^{2} E_i\right)\right] \cup (E_1 \cup E_2)$$

and $E - \left(\bigcup_{i=1}^{2} E_i\right)$ and $(E_1 \cup E_2)$ are disjoint.

Therefore $$vE = v\left[E - \left(\bigcup_{i=1}^{2} E_i\right)\right] + v[E_1 \cup E_2]$$

$$\Rightarrow \quad v\left[E - \left(\bigcup_{i=1}^{2} E_i\right)\right] = v(E) - v[E_1 \cup E_2]$$

$$= v(E) - [vE_1 + vE_2]$$

$$= vE - vE_1 - vE_2$$

Since vE_1 and vE_2 are negative, it follows that

$$v\left[E - \left(\bigcup_{i=1}^{2} E_i\right)\right] \geq 0.$$

If $E - \left(\bigcup_{i=1}^{2} E_i\right)$ is positive, we can take $A = E - \left[\bigcup_{i=1}^{2} E_i\right]$ and the Theorem is established. If it is not so, then it contains sets of negative measures, let n_3 be the smallest integer such that there is a measurable set $E_3 \subset \left[E - \left(\bigcup_{i=1}^{2} E_i\right)\right]$ with $vE_3 < -\frac{1}{n_3}$. Proceeding by induction, let n_k be the smallest integer for which there is a measurable set $E_k \subset E - \left[\bigcup_{i=1}^{k-1} E_i\right]$ and

$$vE_k < -\frac{1}{n_k}$$

If we set $A = E - \left[\bigcup_{k=1}^{\infty} E_k\right]$ …(3)

Then as before

$$E = A \cup \left[\bigcup_{k=1}^{\infty} E_k\right]$$

Since this is a disjoint union, we have

$$VE = vA + v\left[\bigcup_{k=1}^{\infty} E_k\right]$$

$$= vA + \sum vE_k$$

$$< vA - \sum_{k=1}^{\infty} \frac{1}{n_k}.$$

Since vE is finite, the series on the R.H.S. converges absolutely. Thus $\sum \frac{1}{n_k}$ converges and we have $n_k \to \infty$. Since $vE_k \leq 0$ and $vE > 0$, we must have $vA > 0$.

It remains to show that A is positive set. Let $\in > 0$ be given. It is clear from (3) that A is the difference of two measurable sets and therefore A is measurable.

Let $\in > 0$ be given. Since $\sum \frac{1}{n_k}$ converges, this implies that $n_k \to \infty$, we may choose k so large that

$$(n_k - 1)^{-1} < \in$$

Since $$A \subset E - \left[\bigcup_{j=1}^{k} E_j\right]$$

A can contain no measurable sets with measure less than $-\frac{1}{n_k - 1}$ which is greater than $-\in$. Thus A contains no measurable sets of measure less than $-\in$. Since $\in$ is an arbitrary positive number, it follows that A can contain no sets of negative measure and so must be a positive set.

Definition :- Hahn Decomposition

A decomposition of X into two disjoint sets A and B such that A is positive with respect the signed measure v and B is negative with respect the signed measure v is called a Hahn Decomposition for the signed measure v.

Hahn Decomposition Theorem

Statement :- Let v be a signed measure on the measurable space (X, B). Then there is a positive set A and a negative set B such that

$$X = A \cup B \text{ and } A \cap B = \phi .$$

Proof :- Let v be a signed measure defined on the measurable space (X, B). By definition v assumes at most one of the values $+\infty$ and $-\infty$. Therefore w.l.o.g. we may assume that $+\infty$ is the infinite value omitted by v. Let λ be the sup of vA over all sets A which are positive with respect to v. Since the empty set is positive, $\lambda \geq 0$. Let $\{A_i\}$ be a sequence of positive sets such that

$$\lambda = \lim_{i\to\infty} vA_i$$

and set $$A = \bigcup_{i=1}^{\infty} A_i$$

Since countable union of positive sets is positive.

Therefore $$\lambda \geq vA. \quad (1)$$

But $A - A_i \subset A$ and so $v(A-A_i) \geq 0$.

Since $A = (A - A_i) \cup A_i$

$$\Rightarrow \quad vA = v(A - A_i) + v(A_i)$$

$$\geq v(A_i)$$

Hence $$vA \geq \lambda \quad (2)$$

Thus we have from (1) and (2)

$$vA = \lambda.$$

which implies that $vA = \lambda$ and $\lambda < \infty$.

Let $B = A^C$ and let E be a positive subset of B. Then E and A are disjoint and $E \cup A$ is a positive set. Hence

$$\lambda \geq v(E \cup A) = vE + vA$$

$$= vE + \lambda.$$

$\Rightarrow$ $vE = 0$ [where $0 \le \lambda < \infty$]

Thus B contains no positive subset of positive measure and hence no subset of positive measure by the previous Lemma. Consequently B is a negative set and

$$A \cap B = \phi.$$

Remark :- The above theorem states the existence of a Hahn decomposition for each signed measure. Unfortunately, a Hahn-decomposition need not be unique. Infact, it is unique except for null sets. For if $X = A_1 \cup B_1$ and $X = A_2 \cup B_2$ are two Hahn decompositions of X, then we can show that for a measurable set E,

$$v(E \cap A_1) = v(E \cap A_2)$$

and $v(E \cap B_1) = v(E \cap B_2)$

To see this, we observe that

$$E \cap (A_1 - A_2) \subset (E \cap A_1)$$

so that $v[E \cap (A_1 - A_2)] \ge 0$

Moreover $E \cap (A_1 - A_2) \subset E \cap B_2$

$\Rightarrow$ $v[E \cap (A_1 - A_2)] \le 0$

Hence $v[E \cap (A_1 - A_2)] = 0$

and by symmetry

$$v[E \cap (A_2 - A_1)] = 0$$

Thus $v(E \cap A_1) = v[E \cap (A_1 \cup A_2)] = v[E \cap A_2]$

Mutually Singular Measures

Definition :- $v^+(E) = v(E \cap A)$

and $v^-(E) = -v(E \cap B)$

are called respectively positive and negative variations of v. The measure |v| defined by

$$|v|(E) = v^+ E + v^- E$$

is called the absolute value or total variation of v.

Definition :- Two measures v_1 and v_2 on a measurable space (X, B) are said to be mutually singular if there are disjoint measurable sets A and B with X = A $\cup$ B such that

$$v_1(A) = v_2(B) = 0$$

Thus the measures v^+ and v^- defined above are mutually singular since

$$v^+(B) = v(B \cap A) = v(\phi) = 0$$

and $$v^-(A) = -v(A \cap B) = -v(\phi) = 0$$

Jordan Decomposition

Definition :- Let v be a signed measure defined on a measurable space (X, B). Let v^+ and v^- be two mutually singular measures on (X, B) such that $v = v^+ - v^-$. Then this decomposition of v is called **the Jordan Decomposition of v.**

Since v assumes at most one of values $+\infty$ and $-\infty$, either v^+ and v^- must be finite. If they are bot finite, we call v, a finite signed measure. A set E is positive for v if $v^- E = 0$. It is a null set if $|v|(E) = 0$.

Definition :- A measure v is said to be absolutely continuous with respect to measure μ if $vA = 0$ for each set A for which $\mu A = 0$. We use the symbol $v < < \mu$ when v is absolutely continuous with respect to μ.

In the case of signed measures μ are v, we say that $v \leq \mu$ if $|v| < < |\mu|$ and $v \perp \mu$ if

$$|v| \perp |\mu|$$

Definition :- Let μ be a measure and f, a non-negative measurable function on X. For E in B, set

$$vE = \int_E f \, d\mu$$

Then v is a set function defined on B. Also v is countably additive and hence a measure and the measure v will be finite if and only if f is integrable since the integral over a set of μ-measure zero is zero.

Jordan Decomposition Theorem

Proposition :- Let v be a signed measure on a measurable space (X, B). Then there are two mutually singular measures v^+ and v^- on (X, B) such that $v = v^+ - v^-$. Moreover, there is only one such pair of mutually singular measure.

Proof :- Since by definition

$$v^+(E) = v(E \cap A)$$

$$v^-(E) = -v(E \cap B)$$

$$v(E) = v(E \cap A) + v(E \cap B)$$

$$= v^+ - v^-$$

Also v^+ and v^- are mutually singular since

$$v^+B = v(A \cap B) = v(\phi) = 0$$

$$v^- A = -v(B \cap A) = -v(\phi) = 0.$$

where $X = A \cup B$.

Since each such pair determines a Hahn decomposition and also we have. Hahn-decomposition is unique except for null sets. Thus there is only one such pair of mutually singular measures. Also v takes at most one of the values $+\infty$ and $-\infty$ implies that at least one of the set functions v^+ and v^- is always finite.

Radon–Nikodym Theorem

Let (X, B, μ) be a σ-finite measure space and let v be a measure defined on B which is absolutely continuous w.r.t μ. Then there is a non-negative measurable function f such that for each set E in B, we have

$$vE = \int_E f \, d\mu, \quad E \in B.$$

The function f is unique in the sense that if g is any measurable function with this property, then g = f a.e. in X w.r.t μ.

Proof :- We first assume that μ is finite. Then $v-\alpha\mu$ is a signed measure for each rational α. Let (A_α, B_α) be a Hahn-Decomposition for $v - \alpha\mu$ and take $A_0 = X$ and $B_0 = \phi$.

Now since $X = A_\beta \cup B_\beta$, $\quad B_\beta^C = A_\beta$

$$B_\alpha - B_\beta \subseteq B_\alpha \text{ and is negative}$$

$$B_\alpha - B_\beta \subseteq A_\beta \text{ and is positive.}$$

Now $\quad B_\alpha - B_\beta = B_\alpha \cap B_\beta^C = B_\alpha \cap A_\beta$

Thus $(v - \alpha\mu)(B_\alpha - B_\beta) \le 0$

$$(v - \beta\mu)(B_\alpha - B_\beta) \ge 0$$

If $\beta > \alpha \Rightarrow \mu(B_\alpha - B_\beta) = 0$, therefore there is a measurable function f such that for each rational number α, we have $f \ge \alpha$ a.e. on A_α and $f \le \alpha$ a. e on B_α. Since $B_0 = \phi$, we may take f to be non-negative. Let E be an arbitrary set in B and set

$$E_k = E \cap \left[\frac{B_{k+1}}{N} \sim \frac{B_k}{N}\right]$$

$$E_\infty = E \sim \cup \frac{B_k}{N}$$

Then $E = E_\infty \cup [\cup E_k]$

And this is a disjoint union. Hence

$$vE = vE_\infty + v[\cup E_k]$$

$$= v E_\infty + \sum_{k=0}^{\infty} vE_k$$

Since $E_k = \left[\frac{B_{k+1}}{N} - \frac{B_k}{N}\right] \cap E$

Thus $E_k = E \cap \left[\frac{B_{k+1}}{N} \cap \left(\frac{B_k}{N}\right)^C\right]$

$$= E \cap \left[\frac{B_{k+1}}{N} \cap \frac{A_k}{N}\right] \text{ since } B_k^C = A_k$$

Hence $E_k \subset \frac{B_{k+1}}{N} \cap \frac{A_k}{N}$, we have

$$\frac{k}{N} \le f \le \frac{k+1}{N} \text{ on } E_k \text{ from the above existence of f.}$$

and so

$$\frac{k}{N}\mu E_k \le \int_{E_k} f \, d\mu \le \frac{k+1}{N} \mu E_k \quad (1)$$

$$\frac{k}{N}\mu E_k \le vE_k \le \frac{k+1}{N}\mu\, E_k$$

Thus we have

$$\frac{k}{N}\mu\, E_k \le v\, E_k \qquad (2)$$

and $$vE_k \le \frac{k+1}{N}\mu\, E_k \qquad (3)$$

Now from (2), we have

$$vE_k \ge \frac{k}{N}\mu\, E_k$$

$\Rightarrow$ $$vE_k + \frac{1}{N}\mu\; E_k \ge \frac{k}{N}\mu\, E_k + \frac{1}{N}\mu\, E_k$$

$$= \frac{k+1}{N}\mu\, E_k \ge \int_{E_k} f\, d\mu \qquad \text{[from (1)]}$$

Hence

$$vE_k + \frac{1}{N}\mu\, E_k \ge \int_{E_k} f\, d\mu.$$

or $$\int_{E_k} f\; d\mu \le v\, E_k + \frac{1}{N}\;\mu\, E_k \qquad (4)$$

Similarly from (3), we have

$$vE_k \le \frac{k+1}{N}\mu\, E_k$$

$\Rightarrow$ $$vE_k - \frac{1}{N}\mu\, E_k \le \frac{k+1}{N}\;\mu\, E_k - \frac{1}{N}\mu\, E_k$$

or $$vE_k - \frac{1}{N}\;\mu\, E_k \le \frac{k}{N}\mu\, E_k \le \int_{E_k} f\; d\mu \qquad \text{[from (1)]}$$

Thus $$vE_k - \frac{1}{N}\;\mu\, E_k \le \int_{E_k} f\; d\mu \qquad (5)$$

Combining (4) and (5) we have

$$vE_k - \frac{1}{N}\ \mu\, E_k \leq \int_{E_k} f\ d\mu \leq vE_k + \frac{1}{N}\ \mu\, E_k. \tag{6}$$

On E_∞, we have $f = \infty$ a.e.

If $\mu\, E_\infty > 0$, we must have $v\, E_\infty = \infty$.

Since $(\gamma - \alpha\mu)\, E_\infty$ is positive for each α.

If $\mu\, E_\infty = 0$, we must have $vE_\infty = 0$ since $v << \mu$. In either case, we can write

$$vE_\infty = \int_{E_\infty} f\, d\mu \tag{7}$$

Thus from (6) and (7), we have

$$vE - \frac{1}{N}\mu E \leq \int_E f\, d\mu \leq vE + \frac{1}{N}\mu\, E$$

Since $\mu\, E$ is finite and N arbitrary, we must have

$$vE = \int_E f\, d\mu.$$

To show that the theorem is proved for σ-finite measure μ, decompose X into countable union of X_i′s of finite measure. Applying the same argument for each X_i, we get the required function f.

To show the second part, let g be any measurable function satisfying

$$vE = \int_E g\, du, \quad E \in B$$

For each $n \in N$

define $$A_n = \left\{ x \in X, f(x) - g(x) \geq \frac{1}{n} \right\} \in B$$

and $$B_n = \left\{ x \in X, g(x) - f(x) \geq \frac{1}{n} \right\} \in B$$

Since $$f(x) - g(x) \geq \frac{1}{n}\ \forall \quad x \in A_n$$

$\Rightarrow$ $$\int_{A_n} (f - g)d\mu \geq \frac{1}{n}\mu\, A_n$$

By linearity, we have

$$\int_{A_n} f\, d\mu - \int_{A_n} g\, d\mu \geq \frac{1}{n}\mu A_n$$

$\Rightarrow$ $$v A_n - v A_n \geq \frac{1}{n}\mu A_n$$

$\Rightarrow$ $$0 \geq \frac{1}{n}\mu A_n$$

$\Rightarrow$ $$\mu A_n \leq 0$$

Since μA_n can not be negative, we have

$$\mu A_n = 0.$$

Similarly we can show that

$$\mu B_n = 0$$

If we take

$$C = \{x \in X, f(x) \neq g(x)\} = \cup \{A_n \cup B_n\}$$

But $\mu A_n = 0 = \mu B_n$

$\Rightarrow$ $$\mu C = \mu A_n + \mu B_n = 0$$

$\Rightarrow$ $$\mu C = 0$$

Hence f = g a.e w.r.t. measure μ.

Remark :- The function f given by above theorem is called Radon-Nikodym derivative of v with respect to μ. It is denoted by $\left[\frac{dv}{du}\right]$

Lebesgue Decomposition Theorem

Let (X, B, μ) be a σ-finite measure space and v a σ-finite measure defined on B. Then we can find a measure v_0 which is singular w.r.to μ and a measure v_1 which is absolutely continuous with respect to μ such that $v = v_0 + v_1$ where the measures v_0 and v_1 are unique.

Proof :- Since μ and v are σ-finite measures, so is the measure $\lambda = \mu + v$. Since both μ and v are absolutely continuous with respect to λ. Then Radon-

Nikodym theorem asserts the existence of non-negative measurable functions f and g such that for each $x \in E \in B$

$$\mu E = \int_E f \, d\lambda, \quad v E = \int_E g \, d\lambda.$$

Let A = {x ; f(x) >0} and B = {x ; f(x) = 0}. Then X is the disjoint union of A and B while $\mu B = 0$. If we define v_0 by

$$v_0 E = v(E \cap B)$$

Then $\quad v_0 A = v(A \cap B) = v(\phi) = 0$

Since A and B are disjoint.

and so $\quad v_0 \perp \mu.$

Let $\quad v_1(E) = v(E \cap A) = \int_{E \cap A} g \, d\lambda$

$$v_0 E + v_1 E = v(E \cap B) + v(E \cap A)$$

$$= v[(E \cap B) \cup (E \cap A)]$$

$$= v[E \cap (A \cup B)]$$

$$= v[E \cap X]$$

$$= v(E)$$

Thus $\quad v = v_0 + v_1$

It remains to show that $v_1 << \mu$. Let E be a set of μ-measure zero. Then

$$0 = \mu E = \int_E f \, d\lambda.$$

and therefore f = 0 a.e. w.r.t. λ on E.

Since f > 0 on $A \cap E$, we must have

$$\lambda(A \cap E) = 0. \text{ Hence } v(A \cap E) = 0$$

and so $\quad v_1 E = v(A \cap E) = 0$

$\Rightarrow \quad v_1 << \mu$

Thus v_1 is absolutely continuous w.r.t. μ

Now to prove the uniqueness of v_0 and v_1, let v_0' and v_1' be measures such that $v = v_0' + v_1'$ which has the same properties as that of measures v_0 and v_1. Then

$v = v_0 + v_1$ and $v = v_0' + v_1'$ are two Lebesgue decomposition of v.

Thus $\quad v_0 - v_0' = v_1' - v_1$.

Taking the union of the support sets of v_0 and v_0', we have a set E_0 such that

$$(v_0 - v_0')(E) = (v_0 - v_0')(E \cap E_0) \text{ and } \mu(E_0) = 0$$

But $v_1' - v_1$ is absolutely continuous w.r.t. μ and therefore zero on E_0 since $\mu E_0 = 0$. Thus for any measurable set E, we have

$$\begin{aligned}(v_1' - v_1)E &= (v_0 - v_0')E \\ &= (v_0 - v_0')(E \cap E_0) \\ &= (v_1' - v_1)(E \cap E_0) \\ &= 0\end{aligned}$$

since $v_1' - v_1$ is zero on E_0

Thus $\quad v_0 E = v_0' E$

and $\quad v_1' E = v_1 E$

for all measurable sets E which proves the uniqueness of v_0 and v_1.

Remark :- The identity $v = v_0 + v_1$ provided by the preceding theorem (where v_0 is singular w.r.t. μ and v_1 is absolutely continuous with respect to μ) is called the Lebesgue Decomposition of v with respect to μ.

Lebesgue-Stieltjes Integral

Let X be the set of real numbers and B the class of Borel sets. A measure μ defined on B and finite for bounded sets is called a Baire measure (on the real line) to each finite Baire measure, we associate a function F by setting.

$$F(x) = \mu(-\infty, x]$$

The function F is called the cumulative distribution function of μ and is real valued and monotone increasing we have

$$\mu(a, b] = F(b) - F(a)$$

Since (a, b] is the intersection of the sets

$$\left(a, b + \frac{1}{n}\right]$$

$\Rightarrow$ $$\mu\,(a, b] \equiv \lim_{n\to\infty} \mu\left(a, b + \frac{1}{n}\right]$$

and so

$$F(b) - F(a) = \lim_{n\to\infty} \left[F\left(b + \frac{1}{n}\right) - F(a)\right]$$

$\Rightarrow$ $$F(b) = \lim_{n\to\infty} F\left(b + \frac{1}{n}\right) = F(b\,+)$$

Thus a cumulative distribution function continuous on the right. Similarly

$$\mu(b) = \lim_{n\to\infty} \mu\left(b - \frac{1}{n}, b\right]$$

$$= \lim_{n\to\infty} \left[F(b) - F\left(b - \frac{1}{n}\right)\right]$$

$$= F(b) - F(b\,-)$$

Hence F is continuous at b iff the set {b} consisting of b alone has measure zero.

Since $\phi = \cap\,(-\infty, -n]$

$\Rightarrow$ $$\mu\,\phi = \lim_{n\to\infty} \mu(-\infty, -n]$$

$$0 = \lim_{n\to\infty} [F(-n)]$$

$\Rightarrow$ $$\lim_{n\to-\infty} F(n) = 0$$

$\Rightarrow$ $$\lim_{x\to-\infty} F(x) = 0.$$

Since F is monotonic.

Thus we have proved that if μ is finite Baire measure on the real line, then its Commulative Distribution function F is a monotone increasing bounded function which is continuous on the right.

$$\text{and} \qquad \lim_{x\to-\infty} F(x) = 0$$

Definition :- If ϕ is a non-negative Borel measurable function and F is a monotone increasing function which is continuous on the right. We define Lebesgue-Stieltjes Integral of ϕ with respect to F as

$$\int \phi \, dF = \int \phi \, d\mu.$$

where μ is the Baire measure having F as it cumulative distribution function. If ϕ is both positive and negative, we say that it is integrable w.r.t F if it is integrable w.r.t. μ.

Definition :- If F is any monotone increasing function then F* is a monotone increasing function defined by

$$F^*(x) = \lim_{y\to x+} F(y)$$

which is continuous on the right and equal to F where ever F is continuous on the right. Also

$(F^*)^* = F^*$ and if F and G are monotone increasing functions wherever they both are continuous, then $F^* = G^*$. Thus there is a unique function F* which is monotone increasing continuous on the right and agrees with F wherever F is continuous on the right. Then we define L-Stieltjes integral of ϕ w.r.t. F by $\int \phi \, dF = \int \phi \, dF^*$.

Proposition :- Let F be a monotone increasing function continuous on the right. If $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i]$. Then

$$F(b) - F(a) \le \sum_{i=1}^{\infty} [F(b_i) - F(a_i)]$$

Proof :- Write $\cup_i = (a_i, b_i)$ and select intervals as follows. Let $a \in \cup_k$, say $b_k \le b$. Let k_2 be such that $b_{k_1} \in \cup_{k_2}$ etc. By the induction, this sequence comes to end when $b_{k_m} > b$. Renuwhereing the intervals, we have chosen

$U_1, U_2, \ldots, U_m$ where

$a_{i+1} < b_i < b_{i+1}, \; i = 1, 2, \ldots, m-1$

$\Rightarrow$ $F(b) - F(a) \leq F(b_m) - F(a_1)$

$$\leq \sum_{i=1}^{m} [F(b_i) - F(a_i)]$$

$$\leq \sum_{i=1}^{\infty} [F(b_i) - F(a_i)]$$

Product Measures

Let (X, S, u) and (Y, Σ, v) be two fixed measure spaces. The product semi-ring $S\times\Sigma$ of subsets of $X\times Y$ is defined by

$$S \times \Sigma = \{A \times B; A\in S \text{ and } B\in\Sigma\}$$

The above collection $S \times \Sigma$ is indeed a semiring of subsets of $X\times Y$.

Now define the set function $u\times v : S\times\Sigma\rightarrow[0, \infty]$ by

$$u\times v\ (A\times B) = u(A).\ \mu(B)$$

for each $A \times B\in S\times\Sigma$.

This set function is a measure on the product semiring $S \times \Sigma$, called the product measure of μ and v. (proof given below)

Theorem :- The set function $u\times v : S\times\Sigma\rightarrow [0, \infty]$ defined by

$$u\times v\ (A\times B) = u(A).\ v(B)$$

for each $A \times B \in S\times\Sigma$ is a measure

Proof :- Clearly $u\times v(\phi) = 0$. For the subadditivity of $u\times v$, let $A\times B\in S\times\Sigma$ and $(A_n \times B_n)$ be a sequence of mutually disjoint sets of $S\times\Sigma$ such that

$$A \times B = \bigcup_{n=1}^{\infty} A_n \times B_n.$$

It must be established that

$$u(A).\ v(B) = \sum_{n=1}^{\infty} u(A_n).\ v(B_n) \qquad \ldots(*)$$

Obviously $(*)$ holds if either A or B has measure zero. Thus we can assume that

$$u(A) \neq 0 \text{ and } v(B) \neq 0.$$

Since $\chi_{A\times B} = \sum_{n=1}^{\infty} \chi_{A_n \times B_n}$, we see that

$$\chi_A(x).\ \chi_B(y) = \sum_{n=1}^{\infty} \chi_{A_n}(x).\chi_{B_n}(y)$$

holds for all x and y. Now fix $y \in B$.

Since $\chi_{B_n}(y)$ equals one or zero, it follows that

$$\chi_A(x) = \sum_{i\in k} \chi_{A_i}(x), \text{where } k = \{i \in N,\ y \in B_i\}$$

observe that the collection $\{A_i;\ i \in k\}$ must be disjoint and thus

$$u(A) = \sum_{i\in k} u(A_i) \text{ holds. Therefore}$$

$$u(A).\ \chi_B(y) = \sum_{n=1}^{\infty} u(A_n).\chi_{B_n}(y) \qquad \ldots(**)$$

holds for all $y \in Y$. Since a term with $\mu(A_n) = 0$ does not alter the sum in $(*)$ or $(**)$, we can assume that $u(A_n) \neq 0$ for all n. Now if both A and B have finite measures, then integrating term by term, we see that $(*)$ holds. On the other hand if either A or B has infinite measure, then

$$\sum_{n=1}^{\infty} u(A_n).\ v(B_n) = \infty$$

must hold. Indeed if the last sum is finite, then $u(A)\chi_B(y)$ defines an integrable function which is impossible. Thus in this case $(*)$ holds with both sides infinite. Hence the result.

The next few results will unveil the basic properties of the product measure $u\times v$. As usual $(u \times v)^*$ denotes the outer measure generated by the measure space $(X \times Y,\ S\times\Sigma,\ u\times v)$ on $X\times Y$.

Theorem :- If $A \subseteq X$ and $B \subseteq Y$ are measurable sets of finite measure, then

$$(u\times v)^*(A \times B) = u^* \times v^*(A \times B) = u^*(A).\ v^*(B)$$

Proof :- Clearly $S\times\Sigma \subseteq \Lambda_u \times \Lambda_v$ holds. Now let $\{A_n \times B_n\}$ be a sequence of $S\times\Sigma$ such that $A \times B \subseteq \bigcup_{n=1}^{\infty} (A_n \times B_n)$. Since by the last theorem, $u^* \times v^*$ is a measure on the semiring $\Lambda_u \times \Lambda_v$, it follows that

$$u^* \times v^*(A \times B) \le \sum_{n=1}^{\infty} u^* \times v^* (A_n \times B_n)$$

$$= \sum_{n=1}^{\infty} u \times v (A_n \times B_n)$$

and so

$$u^* \times v^* (A \times B) \le (u \times v)^* (A \times B)$$

On the other hand, if $\in >0$ is given, choose two sequences $\{A_n\} \subseteq S$ and $\{B_n\} \subseteq \Sigma$ with $A \subseteq \bigcup_{n=1}^{\infty} A_n$, $B \subseteq \bigcup_{n=1}^{\infty} B_n$ such that

$$\sum_{n=1}^{\infty} u(A_n) < u^*(A) + \in \text{ and}$$

$$\sum_{n=1}^{\infty} v(B_n) < v^*(B) + \in.$$

But then $A \times B \subseteq \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \times B_m$ holds and so

$$(u \times v)^* (A \times B) \le \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u \times v (A_n \times B_m)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u(A_n).\, v(B_m)$$

$$= \left[\sum_{n=1}^{\infty} u(A_n)\right] . \left[\sum_{m=1}^{\infty} v(B_m)\right]$$

$$< [u^*(A) + \in].\, [v^*(B) + \in]$$

for all $\in >0$, that is

$$(u \times v)^* (A \times B) \le u^*(A).\, v^*(B) = u^* \times v^*(A \times B)$$

Therefore

$$(u \times v)^* (A \times B) = u^* \times v^* (A \times B) \text{ holds as required.}$$

Theorem :- If A is a u-measurable subset of X and B, a v-measurable subset of Y, then $A \times B$ is a $u \times v$ measurable subset of $X \times Y$.

Proof :- Let $C \times D \in S \times \Sigma$ with

$$u\times v(C\times D) = u(C).\, v(D) < \infty.$$

To establish $u\times v$ measurability of $A\times B$, it is enough to show that

$$(u\times v)^* \,((C\times D) \cap (A\times B)) + (u \times v)^* \,((C\times D) \cap (A\times B)^C) \le u\times v\,(C\times D)$$

If $u\times v\,(C\times D) = 0$, then the above inequality is obvious. So we can assume $u(C) < \infty$ and $v(D)<\infty$. Clearly

$$(C\times D) \cap (A\times B) = (C \cap A) \times (D \cap B)$$

$$(C\times D) \cap (A\times B)^C = [(C \cap A^C) \times (D\cap B)] \cup [(C \cap A) \times (D\cap B^c)]$$

$$\cup\, [(C \cap A^C) \times (D \cap B^C)]$$

hold with every member of the above union having finite measure. Now the subadditivity of $(u\times v)^*$ combined with the last theorem gives

$$(u\times v)^* \,((C\times D) \cap (A\times B)) + (u \times v)^* \,((C\times D) \cap (A \times B)^C)$$

$$\le u^*(C \cap A).\, v^*(D \cap B) + u^*(C \cap A^C).\, v^* (D\cap B)$$

$$+\, u^*(C \cap A).\, v^*(D \cap B^C) + u^*(C \cap A^C).\, v^*(D \cap B^C)$$

$$= [u^* (C \cap A) + u^*(C \cap A^C)].\, [v^*(D \cap B) + v^*(D \cap B^C)]$$

$$= u(C).\, v(D)$$

$$= u\times v\,(C \times D)$$

as required.

Remark :- In general, it is not true that the measure $u^* \times v^*$ is the only extension of $u\times v$ from $S \times \Sigma$ to a measure on $\Lambda_u \times \Lambda_v$. However if both (X, S, u) and (Y, Σ, v) are σ-finite measure spaces, then $(X \times Y, S \times \Sigma, u\times v)$ is likewise a σ-finite measure space, and therefore $u^* \times v^*$ is the only extension of $u\times v$ to a measure on $\Lambda_u \times \Lambda_v$. Moreover since $\Lambda_u \times \Lambda_v \subseteq \Lambda_{u\times v}$ and the fact that $(u \times v)^*$ is a measure on $\Lambda_{u\times v}$, it follows in this case that $(u \times v)^* = u^* \times v^*$ holds on $\Lambda_u \times \Lambda_v$.

Definition :- If A is a subset of $X\times Y$, and $x\in X$, then the x-section of A is defined by

$$A_x = \{y\in Y \,;\, (x, y)\in A\}$$

Clearly A_x is a subset of Y. Similarly if $y \in Y$, then the y-section of A is defined by

$$A_y = \{x \in X; (x, y) \in A\}$$

Clearly A_y is a subset of Y.

Remark :- The following theorem shows that the relation between the $u \times v$ measurable subsets of $X \times Y$ and the measurable subsets of X and Y.

Theorem :- Let E be a $u \times v$ measurable subset of $X \times Y$ with $(u \times v)^* (E) < \infty$. Then for u-almost all x, the set E_x is a v-measurable subset of Y, and the function $X \to v^*(E_x)$ defines an integrable function over X such that

$$(u \times v)^* (E) = \int_X v^*(E_x) \, d\,u(x). \quad \ldots(1)$$

Similarly, for v-almost all y, the set E^y is a u-measurable subset of X and the function $y \to u^* (E^y)$ defines an integrable function over such that

$$(u \times v)^* (E) = \int_Y u^*(E^y) \, dv(y) \quad \ldots(2)$$

Proof :- Due to symmetry of (1) and (2), it is enough to establish the first formula. The proof goes by steps.

Step I :- Assume $E = A \times B \in S \times \Sigma$. Clearly $E_x = B$ if $x \in A$ and $E_x = \phi$ if $x \notin A$. Thus E_x is a v-measurable subset of Y for each $x \in X$ and

$$v(E_x) = v(B) \, \chi_A(x) \quad \ldots(3)$$

holds for all $x \in X$.

Since $(u \times v)^* (E) = (u \times v) (A \times B) = u(A) \,.\, v(B) < \infty$, two possibilities arise :

(a) Both A and B have finite measure. In this case (3) shows that $x \to v^*(E_x)$ is an integrable function (actually, it is a step function). Such that

(b) $\int_X v^*(E_x) \, d\,u(x) = \int v(B) \, \chi_A \, du = u(A).\, v(B) = (u \times v)^* (E).$

(c) Either A or B has infinite measure.

In this case, the other set must have measure zero and so (3) shows that $v(E_x) = 0$ for u-almost all x. Thus $x \to v^*(E_x)$ defines the zero function and therefore

$$\int_X v^*(E_x)\, du(x) = 0 = (u \times v)^*(E)$$

Step II :- Assume that E is a σ-set of $S \times \Sigma$. Choose a disjoint sequence $\{E_n\}$ of $S \times \Sigma$ such that $E = \bigcup_{n=1}^{\infty} E_n$. In view of $E_x = \bigcup_{n=1}^{\infty} (E_n)_x$ and he preceding step, it follows that E_x is a measurable subset of Y for each $x \in X$. Now define $f(x) = v^*(E_x)$ and

$f_n(x) = \sum_{i=1}^{n} v((E_i)_x)$ for each $x \in X$ and all n. By step I, each f_n defines an integrable function and

$$\int f_n\, du = \sum_{i=1}^{n} \int_X v((E_i)_x)\, d\, u(x)$$

$$= \sum_{i=1}^{n} u \times v(E_i) \uparrow (u \times v)^*(E) < \infty$$

Since $\{(E_n)_x\}$ is a disjoint sequence of Σ, we have

$$v^*(E_x) = \sum_{n=1}^{\infty} v((E_n)_x) \text{ and so } f_n(x) \uparrow f(x).,$$

holds for each $x \in X$. Thus by Levi's theorem "Assume that a sequence $\{f_n\}$ of integrable functions satisfies $f_n \le f_{n+1}$ a.e. for all n and $\lim \int f_n\, du < \infty$. Then there exists an integrable function f such that $f_n \uparrow f$ a.e. and hence $\int f_n\, du \uparrow \int f\, du$ holds" f defines an integrable function and

$$\int_X v^*(E_x)\, dux = \int f\, du = \lim \int f_n\, du$$

$$= \sum_{i=1}^{\infty} u \times v(E_i) = (u \times v)^*(E)$$

Step III :- Assume that E is a countable intersection of σ-sets of finite measure. Choose a sequence $\{E_n\}$ of σ-sets such that $E = \bigcap_{n=1}^{\infty} E_n$,

$$(u \times v)^*(E_1) < \infty \text{ and } E_{n+1} \subseteq E_n \text{ for all n.}$$

For each n, let $g_n(x) = 0$ if

$v^*((E_n)_x) = \infty$ and $g_n(x) = v^*((E_n)_x)$ if

$v^*((E_n)_x) < \infty$. By step II, each g_n is an integrable function over X such that $\int g_n\, du = (u \times v)^*(E_n)$ holds. In view of $E_x = \bigcap_{n=1}^{\infty} (E_n)_x$, it follows that E_x is a v-measurable set for each $x \in X$. Also since $v^*((E_1)_x) < \infty$ holds for u-almost all x, it follows that $g_n(x) = v^*((E_n)_x) \downarrow v^*(E_x)$ holds for u-almost all x. Thus $x \to v^*(E_x)$ defines an integrable function and

$$\int_X v^*(E_x)\, du(x) = \lim \int g_n\, du = \lim\, (u \times v)^*(E_n) = (u \times v)^*(E)$$

Step IV :- Assume that $(u \times v)^*(E) = 0$, thus there exists a measurable set G, which is a countable intersection of σ-sets of finite measure such that $E \subseteq G$ and $(u \times v)^*(G) = 0$. By step III,

$$\int_X v^*(G_x)\, du(x) = (u \times v)^*(G) = 0$$

and so $v^*(G_x) = 0$ holds for u-almost all x. In view of $E_x \subseteq G_x$ for all x, we must have $v^*(E_x) = 0$ for u-almost all x. Therefore E_x is v-measurable for u-almost all x and $x \to v^*(E_x)$ defines the zero function. Thus

$$\int_X v^*(E_x)\, du(x) = 0 = (u \times v)^*(E).$$

Step V :- The general case. Choose a $u \times v$ measurable set F that is a countable intersection of σ-sets all of finite measure such that $E \subseteq F$ and

$(u \times v)^*(F) = (u \times v)^*(E)$. Set $G = F \sim E$.

Then G is a null set and thus by step IV, $v^*(G_x) = 0$ holds for u-almost all x. Therefore E_x is v-measurable and $v^*(E_x) = v^*(F_x)$ holds for u-almost all x. By step III $x \to v^*(F_x)$ defines an integrable function and so $x \to v^*(E_x)$ defines an integrable function and

$$(u \times v)^*(E) = (u \times v)^*(F) = \int_X v^*(F_x)\, d\,u(x)$$

$$= \int_X v^*(E_x)\, d\,ux.$$

holds. The proof of the theorem is now complete.

Definition :- Let $f : X \times Y \to R$ be a function. Then the iterated integral $\iint f\, du\, dv$ is said to exist if f^y is an integrable function over X for v-almost all y and the function $g(y) = \int f^y \, du = \int_X f(x, y) \, du(x)$ defines an integrable function over Y.

If E is a u×v measurable subset of X × Y with $(u \times v)^*(E) < \infty$, then both iterated integrals $\iint \chi_E \, du \, dv$ and $\iint \chi_E \, dv \, du$ exist and

$$\iint \chi_E \, du \, dv = \iint \chi_E \, dv \, du = \int \chi_E \, d(u \times v)$$
$$= (u \times v)^*(E)$$

Since every u×v step function is a linear combination of characteristic functions of u×v measurable sets of finite measure, it follows that if ϕ is a u×v step function, then both iterated integrals $\iint \phi \, du \, dv$ and $\iint \phi \, dv \, du$ exist and moreover

$$\iint \phi \, du \, dv = \iint \phi \, dv \, du = \int \phi \, d(u \times v)$$

The above identities regarding iterated integrals are special cases of a more general result known as Fubini's theorem.

Fubini's Theorem

Let $f : X \times Y \to R$ be u×v integrable function. Then both iterated integrals exist and

$$\iint f d(u \times v) = \iint f du \, dv = \iint f \, dv \, du$$

holds.

Proof :- Without loss of generality, we can assume that $f(x, y) \geq 0$ holds for all x. Choose a sequence $\{\phi_n\}$ of step functions such that

$$0 \leq \phi_n(x, y) \uparrow f(x, y) \text{ holds for all x and y.}$$

Thus

$$\int_X \left[\int_Y \phi_n(x, y) \, dv(y)\right] du(x) = \int \phi_n \, (u \times v) \uparrow fd(u \times v) < \infty \quad \ldots(1)$$

Now by the last theorem, for each n, the function

$$g_n(x) = \int (\phi_n)_x \, dv = \int_Y \phi_n(x, y) \, dv(y)$$

defines an integrable function over X and clearly $g_n(x) \uparrow$ holds for u-almost all x. But then by Levi's Theorem "Assume that a sequence $\{f_n\}$ of integrable functions satisfies $f_n \leq f_{n+1}$ a.e. for all n and lim $\int f_n\, du < \infty$. Then there exists an integrable function f such that $f_n \uparrow f$ a.e.", there exists a u-integrable function $g : X \rightarrow R$ such that $g_n(x) \uparrow g(x)$ u a.e. holds, that is there exists a u-null subset A of X such that $\int(\phi_n)_x\, dv \uparrow g(x)<\infty$ holds for all $x \notin A$. Since $(\phi_n)_x \uparrow f_x$ holds for each x, it follows that f_x is v-integrable for all $x \notin A$ and

$$g_n(x) = \int(\phi_n)_x\, dv = \int_Y \phi_n(x, y)\, dv(y) \uparrow \int_Y f_x\, dv$$

holds for all $x \notin A$.

Now (1) implies that the function $x \rightarrow \int_Y f_x\, dv$ defines an integrable function such that

$$\int f\, d(u \times v) = \int_X \left(\int_Y f_x\, dv \right) du = \iint f\, dv\, du$$

Similarly, $\int f\, d(u \times v) = \iint f du.\, dv$ and the proof of the theorem is complete.

Remark :- The existence of the iterated integrals is by no means enough to ensure that the function is integrable over the product space. As an example of this sort, consider X = Y = [0, 1] u = v = λ (the Lebesgue measure) and

$$f(x, y) = \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \text{ if } (x, y) \neq (0, 0) \text{ and}$$

$$f(0, 0) = 0.$$

Then

$$\iint f\, du\, dv = -\frac{\pi}{4} \text{ and } \iint f\, dv\, du = \frac{\pi}{4}$$

Fubini's theorem shows of course that f is not integrable over $[0, 1] \times [0, 1]$

There is a converse to Fubini's theorem however according to which the existence of one of the iterated integrals is sufficient for the integrality of the function over the product space. This result is known as Tonell's Theorem and this result is frequently used in applications.

Measure and Topology

We are often concerned with measures on a set X which is also a topological space and it is natural to consider conditions on the measure so that it is connected with the topological structure. There seem to be two classes of topological spaces for which it is possible to carry out a reasonable theory. One is the class of locally compact Hausdorff spaces and other is the class of complete metric spaces. The present chapter develops the theory for the class of locally compact Hausdorff spaces.

Baire Sets and Borel Sets

Let X be a locally compact Hausdorff space. Let $C_c(X)$ be the family consisting of all continuous real-valued functions that vanish outside a compact subset of X. If f is a real valued function, the support of f is the closure of the set $\{x ; f(x) \neq 0\}$. Thus $C_c(X)$ is the class of all continuous real valued functions on X with compact support. The class of Baire sets is defined to be the smallest σ-algebra B of subsets of X such that each function in $C_c(X)$ is measurable with respect to B. Thus B is the σ-algebra generated by the sets $\{x; f(x) \geq \alpha\}$ with $f \in C_c(X)$. If $\alpha > 0$, these sets are compact G_δ's. Thus each compact G_δ is a Baire set. Consequently B is the σ-algebra generated by the compact G_γ′s

If X is any topological space, the smallest σ-algebra containing the closed sets is called the class of Borel sets. Thus if X is locally compact, every Baire set is a Borel set. The converse is true when X is a locally compact separable metric space, but there are compact spaces where the class of Borel sets is larger than the class of Baire sets.

Baire Measure

Let X be a locally compact Hausdorff space. By a Baire measure on X, we mean a measure defined for all Baire sets and finite for each compact Baire set. By a Borel measure, we mean a measure defined on the σ-algebra of Borel sets or completion of such a measure.

Definition :- A set E in a locally compact Hausdorff space is said to be (topologically) bounded if E is contained in some compact set i.e. $\overline{E}$ is a compact. A set E is said to be σ-bounded if it is the union of a countable collection of bounded sets. From now onwards, X will be a locally compact Hausdorff space.

Now we state a number of Lemmas that are useful in dealing with Baire and Borel sets.

Lemma 1 :- Let K be a compact set, O an open set with $K \subset O$. Then

$$K \subset \cup \subset H \subset O$$

where U is a σ-compact open set and H is a compact G_δ.

Lemma 2:- Every σ-compact open set is the union of a countable collection of compact G_δ′s and hence a Baire set.

Lemma 3 :- Every bounded set is contained in a compact G_δ. Every σ-bounded set E is contained in a σ-compact open set O. If E is bounded, we may take $\overline{O}$ to be compact.

Lemma 4 :- Let R be a ring of sets and let $R' = \{E; \overline{E} \in R\}$. Then either $R = R'$ or else $R \cap R' = 0$. In the latter case $R \cup R'$ is the smallest algebra containing R. If R is a σ-ring, then $R \cup R'$ is a σ-algebra.

Lemma 5 :- If E is a Baire set, then E or $\overline{E}$ is σ-bounded. Both are σ-bounded if and only if X is σ-compact.

Lemma 6 :- The class of σ-bounded Baire sets is the smallest σ-ring containing the compact G_δ′s.

Lemma 7 :- Each σ-bounded Baire set is the union of a countable disjoint union of bounded Baire sets.

Remark :- The following Proposition gives useful means of applying theorems about Baire and Borel sets in compact spaces to bounded Baire and Borel sets in locally compact spaces.

Proposition :- Let F be a closed subset of X. Then F is a locally compact Hausdorff space and the Baire sets of F are those sets of the form $B \cap F$, where B is a Baire set in X. Thus if F is a closed Baire set, the Baire subsets of F are just those Baire subsets of X which are contained in F. The Borel sets of F are those Borel sets of X which are contained in F.

Proof :- Let

$R = \{ E ; E = B \cap F; B \in Ba(X)\}$ where Ba(X) is the class of Baire sets. Then R is a σ-algebra which includes all compact G_δ′s contained in F. Thus $Ba(F) \subset R$ and each Baire set of F is of the form $B \cap F$. Let

$$B = \{E \subset X ; E \cap F \in Ba(F)\}. \text{ Then}$$

B is a σ-algebra. Let K be a compact G_δ in X . Then $K \cap F$ is a closed subset of K and hence compact. Since K is a G_δ in x, $K \cap F$ is a G_δ in F. Thus $K \cap F$ is a compact G_δ of F and so is in Ba(F). Consequently $Ba(X) \subset B$ and so each Baire set of X interests F in a Baire set of F.

If F is a closed Baire subset of X, then $B \cap F$ is a Baire subset of X whenever B is. Thus each Baire subset of F is of this form. On the other hand for each Baire subset B of X with $B \subset F$ we have $B = B \cap F$ and so B is a Baire subset of F.

Continuous Functions with Compact Support

Let X be a locally compact topological space. If $\phi : X \to R$ and $S = \{x \in X; \phi(x) \neq 0\}$. Then the closure K of S is called the support of ϕ. Suppose that ϕ has support K where K is a compact subset of X. Then ϕ vanishes outside S. Conversely if ϕ vanishes outside some compact set C and $S \subset C$ as C is closed, the closure K of S is contained in C, now K is a closed subset of the compact set C and as such K is compact. Then ϕ has compact support.

Theorem :- Let X be a locally compact Hausdorff space, A and B non-empty disjoint subsets of X, A closed and B compact. Then there is a continuous function $\psi : X \to [0, 1]$ C of compact support such that $\psi(x) = 0$ for all x in A and

$$\psi(x) = 1 \text{ for all x in B.}$$

First we give some Lemma.

Lemma 1 :- Let X be a Hausdorff space, K a compact subset and $p \in K^c$. Then there exists disjoint open subsets G, H such that $p \in G$ and $K \subset H$.

Proof :- To any point x of K, there exist disjoint open sets A_x, B_x such that $p \in A_x$, $x \in B_x$. From the covering $\{B_x\}$ of K, there is a finite subcovering $Bx_1, Bx_2, \ldots, Bx_n$ and the sets

$$G = \bigcap_{i=1}^{n} A_{x_i},\ H = \bigcup_{i=1}^{n} B_{x_i}$$

satisfy the required conditions.

Lemma 2 :- Let X be a locally compact Hausdorff space, K a compact subset, U an open subset and $K \subset U$. Then there exists an open subset V with compact closure $\overline{V}$ such that

$$K \subset V \subset \overline{V} \subset U\,.$$

Proof :- Let G be the open set with compact closure $\overline{G}$. If U = X, we simply take V = G. In general G is too large, the open set $G \cap U$ is compact as its closure is a subset of $\overline{G}$ but its closure may still contain points outside U.

We assume that the complement F of U is not empty. To any point p of F, there are disjoint open sets G_p, H_p such that $p \in G_p$, $K \subset H_p$. As $F \cap \overline{G}$ is compact, there are points $p_1, p_2, \ldots, p_n$ in F such that $G_{p_1}, G_{p_2}, \ldots, G_{p_n}$ cover $F \cap \overline{G}$. We now verify at once that the open set $V = G \cap H_{p_1} \cap \ldots \cap H_{p_n}$ satisfy the conditions of the Lemma 2.

Proof of the theorem :- Let U be the complement of A. According to Lemma 2, there is an open set $V_{1/2}$ with compact closure such that

$$B \subset V_{\frac{1}{2}} \subset \overline{V}_{\frac{1}{2}} \subset U,$$

and then there are open sets $V_{\frac{1}{4}}, V_{\frac{3}{4}}$ with compact closure such that

$$B \subset V_{\frac{3}{4}} \subset \overline{V}_{\frac{3}{4}} \subset V_{\frac{1}{2}} \subset \overline{V}_{\frac{1}{2}} \subset V_{\frac{1}{4}} \subset \overline{V}_{\frac{1}{4}} \subset U$$

Continuing in this way we obtain open sets V_r for each dyadic rational number $r = p/2^m$ in (0, 1) such that

$$B \subset V_r \subset \overline{V}_r \subset U$$

and

$$\overline{V}_r \subset V_s \text{ for } r > s. \qquad \ldots(1)$$

Now we must construct a continuous function $\psi : X \to [0, 1]$. To this end, define for each $r = p/2^m$ in (0, 1).

$$\psi_r(x) = r \text{ if } x \in V_r.$$
$$= 0 \text{ otherwise}$$

and then $\psi = \sup\limits_r \psi_r$.

It follows at once that $0 \le \psi \le 1$, that $\psi = 0$ on A and $\psi = 1$ on B. It follows that ψ_r and ψ are lower semicontinuous. To prove that ψ is continuous, we introduce the upper semicontinuous function θ_r and θ defined by

$$\theta_r(x) = 1 \text{ if } x \in \overline{V}_r ,$$
$$= r \text{ otherwise}$$

and $\theta = \operatorname{Inf}\limits_r \theta_r$.

It is sufficient to show that $\psi = \theta$.

We can only have $\psi_r(x) > \theta_s(x)$ if $r > s$ $x \in V_r$ and $x \notin \overline{V}_s$. But this is impossible by (1), whence $\psi_r \le \theta_s$ for all r, s and so $\psi \le \theta$.

On the other hand, suppose that $\psi(x) < \theta(x)$ then there are dyadic rationals r, s in (0, 1) such that

$$\psi(x) < r < s < \theta(x) .$$

As $\psi(x) < r$, we have $x \notin V_r$ and as $\theta(x) > s$ we have $x \in \overline{V}_s$ which again contradicts (1). Thus $\psi \ge \theta$, combining these inequalities gives $\psi = \theta$ and establishes the continuity of ψ.

Hence the result.

Regularity of Measure

Let μ be a measure defined on a σ-algebra M of subsets of X where X is a locally compact Hausdorff space. and suppose that M contains the Baire sets. A set $E \in M$ is said to be outer regular for μ(or μ is outer regular for E) if

$$\mu E = \operatorname{Inf} \{\mu O : E \subset O, \text{ open}, O \in M\}$$

It is said to be inner regular if

$$\mu E = \sup \{\mu K : K \subset E, K \text{ compact}, K \in M\}$$

The set E is said to be regular for μ if it is both inner and outer regular for μ.

We say that the measure μ is inner regular (outer regular, regular) if it is inner regular (outer regular, regular) for each set $E \in M$. Lebesgue measure is a regular measure.

For compact spaces X, there is complete symmetry between inner regularity and outer regularity. A measurable set E is outer regular if and only if its complement is inner regular. A finite measure on X is inner regular if and only if it is outer regular, and hence regular. When X is compact, every Baire measure is regular.

Remark :- When X is no longer compact, we lose this symmetry because the complement of an open set need not be compact.

Proposition :- Let μ be a finite measure defined on a σ-algebra M which contains all the Baire sets of a locally compact space X. If μ is inner regular, it is regular.

Proof :- Let $E \in M$, then

$$\mu \overline{E} = \sup \{\mu K; K \subset \overline{E}, K \in M \text{ and } K \text{ compact}\}.$$

But for each such a K, we have $\overline{K}$ open and $E \subset \overline{K}$. Hence

$$\begin{aligned} \mu E = \mu X - \mu \overline{E} &= \text{Inf}\{\mu X - \mu K\} \\ &= \text{Inf}\ \mu \overline{K} \\ &\geq \text{Inf}\ \{\mu O; E \subset O\} \end{aligned}$$

Thus

$$\mu E = \text{Inf}\ \{\mu O : E \subset O;\ O \text{ open and } O \in M\}.$$

Theorem : - Let μ be a Baire measure on a locally compact space X and E a σ-bounded Baire set in X. Then for $\in > 0$,

(i) There is a σ - compact open set O with $E \subset O$ and $\mu(O \sim E) < \in$.

(ii) $\mu E = \sup \{\mu K ; K \subset E, K \text{ a compact } G_\delta\}$.

Proof :- Let R be the class of sets E that satisfy (i) and (ii) for each $\in > 0$. Suppose $E = UE_n$, where $E_n \in R$. Then for each n, there is a σ-compact open set O_n with $E_n \subset O_n$ and

$$\mu(O_n \sim E_n) < 2^{-n} \in. \text{ Then } O = U O_n$$

is again a σ-compact open set with

$$\mu(O \sim E) \subset U (O_n \sim E_n)$$

and so

$$\mu(O \sim E) \leq \sum \mu(O_n \sim E_n) < \in$$

Thus E satisfies (i)

If for some, n, we have $\mu\, E_n = \infty$, then there are compact G_δ′s of arbitrary large finite measure contained in $E_n \subset E$. Hence (ii) holds for E. If $u\, E_n < \infty$ for each n, there is a $K_n \subset E_n$, K_n, a compact G_δ and

$$\mu(E_n \sim K_n) < 2^{-n}\in$$

Then

$$\mu\, E = \sup_N \mu\left(\bigcup_{n=1}^{N} E_n\right)$$

$$\leq \sup_N \mu\left(\bigcup_{n=1}^{N} K_n\right) + \in$$

Thus E satisfies (ii). If E is a compact G_δ, then there is a continuous real valued function ϕ with compact support such that $0 \leq \phi \leq 1$ and $E = \{x\ ;\ \phi(x) = 1\}$. Let $O_n = \{x\ ;\ \phi(x) > 1-1/n\}$. Then O_n is a σ-compact open set with $\overline{O}_n$ compact. Since $\mu O_1 < \infty$, we have $\mu E = \text{Inf}\ \mu\, O_n$. Thus each compact G_δ satisfies (i) and it trivially satisfies (ii)

Let X be compact. Then E satisfies (i) if and only if $\overline{E}$ satisfies (ii) and so the collection R of sets satisfying (i) and (ii) is a σ-algebra containing the compact G_δ′s. Thus R contains all Baire sets and the prop. holds when X is compact.

For an arbitrary locally compact space X and bounded Baire set E, we can take H to be a compact G_δ and U to be a σ-compact open subsets of X such that

$$\overline{E} \subset U \subset H.$$

Then E is a Baire subset of H and so

$$\mu\,(W - E) < \in$$

Since W and U are σ-compact, so is $O = W \cap U$. Thus O is a σ-compact open set with $E \subset O \subset W$.

$$O \sim E \subset W \sim E$$

and

$$\mu(O \sim E) < \in.$$

Thus E satisfies (i). Therefore all bounded Baire sets are in R.

Since R is closed under countable unions and each σ-bounded Baire set is a countable union of bounded Baire sets, we see that every σ-bounded Baire set belongs to R .

Remark :- If we had defined the class of Baire sets to be the smallest σ-ring containing the compact G_δ′s and taken a Baire measure to be defined on this σ-ring, then the above theorem takes the form

"Every Baire measure is regular". If X is σ-compact, the σ-ring and the σ-algebra generated by the compact G_δ's are the same. Hence we have the following corollary.

Corollary :- If X is σ-compact, then every Baire measure on X is regular.

Quasi-Measure :- A measure μ defined on σ-algebra M which contains the Baire sets is said to be quasi-regular if it is outer regular and each open set O $\in$M is inner regular for μ.

A Baire measure on a space which is not σ-compact need not be regular but we can require it to be inner regular or quasi-regular without changing its values on the σ-bounded Baire sets.

Proposition :- Let μ be a measure defined on a σ-algebra M containing the Baire sets. Assume either that μ is quasi-regular or that μ is inner regular. Then for each $E \in M$ with $\mu E < \infty$, there is a Baire set B with

$$\mu(E \,\Delta\, B) = O$$

Proof :- We consider only the quasi-regular case. Let E be a measurable set of finite measure. Since μ is outer regular, we can find a sequence $\langle O_n \rangle$ of open sets with

$$O_n \supset O_{n+1} \supset E$$

and

$$\mu\, O_n < \mu\, E + 2^{-n}$$

Since μ is quasi-regular, there is a compact set $K_m \subset O_m$ with

$$\mu K_m > \mu O_m - 2^{-m}$$

and we may take K_m to be a G_δ set by Lemma 1. Now

$$\mu K_m > \mu O_m - 2^{-m} \geq \mu E - 2^{-m}$$
$$> \mu O_n - 2^{-n} - 2^{-m}$$

Set

$$H_m = \bigcup_{j=m}^{\infty} K_j$$

Then H_m is a Baire set, $H_m \subset O_m \subset O_n$ for $m \geq n$. Also $H_m \supset H_{m+1}$, and

$$\mu\, H_m \geq \mu K_m > \mu O_n - 2^{-n} - 2^{-m}.$$

Let $B = \cap H_m$. Then B is a Baire set, $B \subset O_n$ and

$$\mu B = \lim \mu\, H_m$$

Thus

$$\mu B \geq \mu O_n - 2^{-n},$$

Since $B \subset O_n$ and $E \subset O_n$, we have

$$B \Delta E \subset (O_n \sim B) \cup (O_n \sim E)$$

and so

$$\mu(B \Delta E) \leq \mu(O_n \sim B) + \mu(O_n \sim E)$$
$$< 2^{-n} + 2^{-n} = 2^{-n+1}$$

This is true for any n and so

$$\mu(B \Delta E) = 0.$$

Proposition :- Let $\bar{\mu}$ be a non-negative extended real valued function defined on the class of open subsets of X and satisfying

(i) $\bar{\mu} O < \infty$ if $\bar{O}$ compact.

(ii) $\bar{\mu} O_1 \leq \bar{\mu} O_2$ if $O_1 \subset O_2$.

(iii) $\bar{\mu} (O_1 \cup O_2) = \bar{\mu} O_1 + \bar{\mu} O_2$ if $O_1 \cap O_2 = \phi$.

(iv) $\bar{\mu} (UO_i) \leq \sum \bar{\mu} O_i$

(v) $\bar{\mu} (O) == \sup \{\mu U;\ \bar{U} \subset O, \bar{U}\text{ compact}\}$

Then the set function μ^* defined by

$$\mu^*E = \text{Inf } \{ \bar{\mu} O; E \subset O\}$$

is a topologically regular outer measure.

Proof :- The monotonicity and countable subadditivity of μ^* follow directly from (ii) and (iv) and the definition of μ^*. Also $\mu^*O = \bar{\mu} O$ for O open and so condition (ii) of the definition of regularity follows from hypothesis (iii) of the proposition and the condition (i) from the definition of μ^*. Since $\bar{\mu} O < \infty$ for $\bar{O}$ compact, we have $\mu^* E < \infty$ for each bounded set E.

Riesz-Markov Theorem

Let X be a locally compact Hausdorff space. By $C_c(X)$, we denote as usual, the space of continuous real valued functions with compact support. A real valued linear functional I on $C_c(X)$ is said to be positive if $I(f) \geq 0$ whenever $f \geq 0$. The purpose of the following theorem is to prove that every positive linear functional on $C_c(X)$ is represented by integration with respect to a suitable Borel (or Baire) measure. In particular we have the following theorem :

Statement of Riesz-Markov Theorem

Let X be a locally compact Hausdorff space and I a positive linear functional on $C_c(X)$. Then there is a Borel measure μ on X such that

$$I(f) = \int f\, du$$

For each $f \in C_c(X)$. The measure μ may be taken to be quasi-regular or to be inner regular. In each of these cases it is then unique.

Proof :- For each open set O define $\bar{\mu}$ O by

$$\bar{\mu} O = \sup \{I(f); f \in C_c(X), O \leq f \leq 1, \sup f \subset O\}$$

Then $\bar{\mu}$ is an extended real valued function defined on all open sets and is readily seen to be monotone, finite on bounded sets and to satisfy the regularity (v) of the above Proposition. To see that $\bar{\mu}$ is countably subadditive on open sets, let $O = UO_i$ and let f be any function in $C_c(X)$ with $O \leq f \leq 1$ and $\sup f \subset O$. Thus there are non-negative functions $\phi_1, \phi_2, \ldots, \phi_n$ in $C_c(X)$ with $\sup \phi_i \subset O_i$ and

$$\sum_{i=1}^{n} \phi_i = 1.$$

on sup f. Then $f = \sum \phi_i f$, $o \leq \phi_i f \leq 1$ and

$\sup (\phi_i f) \subset O_i$. Thus

$$If = \sum_{i=1}^{n} I(\phi_i f) \leq \sum_{i=1}^{n} \bar{\mu} O_i$$

$$\leq \sum_{i=1}^{\infty} \bar{\mu} O_i$$

Taking the sup over all such f gives

$$\bar{\mu} O \leq \sum_{i=1}^{\infty} \bar{\mu} O_i$$

and $\bar{\mu}$ is countably subadditive.

If $O = O_1 \cup O_2$ with $O_1 \cap O_2 = \phi$ and $f_i \in C_c(X)$, $0 \leq f_i \leq 1$ and $\sup f_i \subset O_i$, then the function $f = f_1 + f_2$ has $\sup f \subset O$ and $0 \leq f \leq 1$. Thus

$$I f_1 + I f_2 \leq \bar{\mu} O.$$

Since f_1 and f_2 can be chosen arbitrarily, subject to $0 \leq f_i \leq 1$ and $\sup f_i \subset O_i$, we have

$$\bar{\mu} O_1 + \bar{\mu} O_2 \leq \bar{\mu} O,$$

whence

$$\bar{\mu} O_1 + \bar{\mu} O_2 = \bar{\mu} O$$

Thus $\bar{\mu}$ satisfies the hypothesis of the above proposition so $\bar{\mu}$ extends to a quasi-regular Borel measure.

We next proceed to show that If = $\int$ f du for each $f \in C_c(X)$. Since f is the difference of two non-negative functions in $C_c(X)$, it is sufficient to consider $f \geq 0$. By linearity we may also take $f \leq 1$.

Choose a bounded open set O with sup $f \subset O$. Set

$$O_k = \{x;\ n\, f(x) > k-1\}$$

and $O_o = O$. Then $O_{n+1} = \phi$ and $\overline{O}_{k+1} \subset O_k$.

Define $$\phi_k = \begin{cases} 1 & \text{in} \quad O_{k+1} \\ nf(x) - k + 1 & \text{in} \quad O_k - O_{k+1} \\ 0 & \text{in} \quad \overline{O}_k \end{cases}$$

Then $$f = \frac{1}{n}\sum_{k=1}^{n} \phi_k$$

We also have sup $\phi_k \subset \overline{O}_k \subset O_{k-1}$ and

$$\phi_K = 1 \text{ on } O_{k+1}. \text{ Thus}$$

$$\bar{\mu}\, O_{k+1} \leq I\phi_k \leq \bar{\mu}\, O_{k-1}$$

for $k \geq 1$.

Also

$$\bar{\mu}\, O_{k+1} \leq \int \phi_k \, d\bar{\mu} \leq \bar{\mu}\, O_k$$

for $k \geq 1$.

Hence

$$-\mu O_1 \leq \sum_{k=1}^{n} (I\phi_k - \int \phi_k) \leq \bar{\mu}\, O_0 + \bar{\mu}\, O_1$$

Consequently

$$|If - \int f du| \leq \frac{2}{n} \bar{\mu}\, O$$

since n is arbitrary,

$$If = \int f \, d\bar{\mu}\, .$$

Thus there is an inner regular Borel measure μ which agrees with $\bar{\mu}$ on the σ-bounded Borel sets. Since only the values of μ on σ-bounded Baire sets enter into $\int$f du, we have

$$If = \int f \, du.$$

The unicity of $\bar{\mu}$ and μ is obvious.

Measure and Topology

We are often concerned with measures on a set X which is also a topological space and it is natural to consider conditions on the measure so that it is connected with the topological structure. There seem to be two classes of topological spaces for which it is possible to carry out a reasonable theory. One is the class of locally compact Hausdorff spaces and other is the class of complete metric spaces. The present chapter develops the theory for the class of locally compact Hausdorff spaces.

Baire Sets and Borel Sets

Let X be a locally compact Hausdorff space. Let $C_c(X)$ be the family consisting of all continuous real-valued functions that vanish outside a compact subset of X. If f is a real valued function, the support of f is the closure of the set $\{x ; f(x) \neq 0\}$. Thus $C_c(X)$ is the class of all continuous real valued functions on X with compact support. The class of Baire sets is defined to be the smallest σ-algebra B of subsets of X such that each function in $C_c(X)$ is measurable with respect to B. Thus B is the σ-algebra generated by the sets $\{x; f(x) \geq \alpha\}$ with $f \in C_c(X)$. If $\alpha > 0$, these sets are compact G_δ's. Thus each compact G_δ is a Baire set. Consequently B is the σ-algebra generated by the compact G_γ's

If X is any topological space, the smallest σ-algebra containing the closed sets is called the class of Borel sets. Thus if X is locally compact, every Baire set is a Borel set. The converse is true when X is a locally compact separable metric space, but there are compact spaces where the class of Borel sets is larger than the class of Baire sets.

Baire Measure

Let X be a locally compact Hausdorff space. By a Baire measure on X, we mean a measure defined for all Baire sets and finite for each compact Baire set. By a Borel measure, we mean a measure defined on the σ-algebra of Borel sets or completion of such a measure.

Definition :- **A set E in a locally compact Hausdorff space is said to be (topologically) bounded if E is contained in some compact set i.e. $\overline{E}$ is a compact. A set E is said to be σ-bounded if it is the union of a countable collection of bounded sets. From now onwards, X will be a locally compact Hausdorff space.**

Now we state a number of Lemmas that are useful in dealing with Baire and Borel sets.

Lemma 1 :- **Let K be a compact set, O an open set with $K \subset O$. Then**

$$K \subset \cup \subset H \subset O$$

where U is a σ-compact open set and H is a compact G_δ.

Lemma 2:- **Every σ-compact open set is the union of a countable collection of compact G_δ′s and hence a Baire set.**

Lemma 3 :- **Every bounded set is contained in a compact G_δ. Every σ-bounded set E is contained in a σ-compact open set O. If E is bounded, we may take $\overline{O}$ to be compact.**

Lemma 4 :- **Let R be a ring of sets and let R′ = {E; $\overline{E}$ ∈R}. Then either R = R′ or else R ∩ R′ = 0. In the latter case R ∪ R′ is the smallest algebra containing R. If R is a σ-ring, then R ∪ R′ is a σ-algebra.**

Lemma 5 :- **If E is a Baire set, then E or $\overline{E}$ is σ-bounded. Both are σ-bounded if and only if X is σ-compact.**

Lemma 6 :- **The class of σ-bounded Baire sets is the smallest σ-ring containing the compact G_δ′s.**

Lemma 7 :- **Each σ-bounded Baire set is the union of a countable disjoint union of bounded Baire sets.**

Remark :- **The following Proposition gives useful means of applying theorems about Baire and Borel sets in compact spaces to bounded Baire and Borel sets in locally compact spaces.**

Proposition :- **Let F be a closed subset of X. Then F is a locally compact Hausdorff space and the Baire sets of F are those sets of the form B∩F, where B is a Baire set in X. Thus if F is a closed Baire set, the Baire subsets of F are just those Baire subsets of X which are contained in F. The Borel sets of F are those Borel sets of X which are contained in F.**

Proof :- **Let**

R = { E ; E = B∩F; B∈ Ba(X)} where Ba(X) is the class of Baire sets. Then R is a σ-algebra which includes all compact G_δ′s contained in F. Thus Ba(F)⊂R and each Baire set of F is of the form B∩F. Let

B = {E⊂X ; E∩F∈Ba(F)}. Then

B is a σ-algebra. Let K be a compact G_δ in X . Then K∩F is a closed subset of K and hence compact. Since K is a G_δ in x, K∩F is a G_δ in F. Thus K∩F is a compact G_δ of F and so is in Ba(F). Consequently Ba(X) ⊂ B and so each Baire set of X interests F in a Baire set of F.

If F is a closed Baire subset of X, then B∩F is a Baire subset of X whenever B is. Thus each Baire subset of F is of this form. On the other hand for each Baire subset B of X with B⊂ F we have B = B∩F and so B is a Baire subset of F.

Continuous Functions with Compact Support

Let X be a locally compact topological space. If $\phi : X \to R$ and $S = \{x \in X; \phi(x) \neq 0\}$. Then the closure K of S is called the support of ϕ. Suppose that ϕ has support K where K is a compact subset of X. Then ϕ vanishes outside S. Conversely if ϕ vanishes outside some compact set C and $S \subset C$ as C is closed, the closure K of S is contained in C, now K is a closed subset of the compact set C and as such K is compact. Then ϕ has compact support.

Theorem :- **Let X be a locally compact Hausdorff space, A and B non-empty disjoint subsets of X, A closed and B compact. Then there is a continuous function $\psi : X \to [0, 1]$ C of compact support such that $\psi(x) = 0$ for all x in A and**

$$\psi(x) = 1 \text{ for all } x \text{ in } B.$$

First we give some Lemma.

Lemma 1 :- **Let X be a Hausdorff space, K a compact subset and $p \in K^c$. Then there exists disjoint open subsets G, H such that $p \in G$ and $K \subset H$.**

Proof :- **To any point x of K, there exist disjoint open sets A_x, B_x such that $p \in A_x$, $x \in B_x$. From the covering $\{B_x\}$ of K, there is a finite subcovering $Bx_1, Bx_2, \ldots, Bx_n$ and the sets**

$$G = \bigcap_{i=1}^{n} A_{x_i}, \; H = \bigcup_{i=1}^{n} B_{x_i}$$

satisfy the required conditions.

Lemma 2 :- **Let X be a locally compact Hausdorff space, K a compact subset, U an open subset and $K \subset U$. Then there exists an open subset V with compact closure $\overline{V}$ such that**

$$K \subset V \subset \overline{V} \subset U.$$

Proof :- **Let G be the open set with compact closure $\overline{G}$. If U = X, we simply take V = G. In general G is too large, the open set $G \cap U$ is compact as its closure is a subset of $\overline{G}$ but its closure may still contain points outside U.**

We assume that the complement F of U is not empty. To any point p of F, there are disjoint open sets G_p, H_p such that $p \in G_p$, $K \subset H_p$. As $F \cap \overline{G}$ is compact, there are points $p_1, p_2, \ldots, p_n$ in F such that $G_{p_1}, G_{p_2}, \ldots, G_{p_n}$ cover $F \cap \overline{G}$. We now verify at once that the open set $V = G \cap H_{p_1} \cap \ldots \cap H_{p_n}$ satisfy the conditions of the Lemma 2.

Proof of the theorem :- **Let U be the complement of A. According to Lemma 2, there is an open set $V_{1/2}$ with compact closure such that**

$$B \subset V_{\frac{1}{2}} \subset \overline{V}_{\frac{1}{2}} \subset U,$$

and then there are open sets $V_{\frac{1}{4}}, V_{\frac{3}{4}}$ **with compact closure such that**

$$B \subset V_{\frac{3}{4}} \subset \overline{V}_{\frac{3}{4}} \subset V_{\frac{1}{2}} \subset \overline{V}_{\frac{1}{2}} \subset V_{\frac{1}{4}} \subset \overline{V}_{\frac{1}{4}} \subset U$$

Continuing in this way we obtain open sets V_r for each dyadic rational number $r = p/2^m$ in (0, 1) such that

$$B \subset V_r \subset \overline{V}_r \subset U$$

and

$$\overline{V}_r \subset V_s \text{ for } r > s. \quad \ldots(1)$$

Now we must construct a continuous function $\psi : X \to [0, 1]$. To this end, define for each $r = p/2^m$ in (0, 1).

$$\psi_r(x) = r \text{ if } x \in V_r.$$
$$= 0 \text{ otherwise}$$

and then $\psi = \sup_r \psi_r$ **.**

It follows at once that $0 \le \psi \le 1$, that $\psi = 0$ on A and $\psi = 1$ on B. It follows that ψ_r and ψ are lower semicontinuous. To prove that ψ is continuous, we introduce the upper semicontinuous function θ_r and θ defined by

$$\theta_r(x) = 1 \text{ if } x \in \overline{V}_r,$$
$$= r \text{ otherwise}$$

and $\theta = \operatorname{Inf}_r \theta_r$ **.**

It is sufficient to show that $\psi = \theta$.

We can only have $\psi_r(x) > \theta_s(x)$ if $r > s$ $x \in V_r$ and $x \notin \overline{V}_s$. But this is impossible by (1), whence $\psi_r \le \theta_s$ for all r, s and so $\psi \le \theta$.

On the other hand, suppose that $\psi(x) < \theta(x)$ then there are dyadic rationals r, s in (0, 1) such that

$$\psi(x) < r < s < \theta(x) .$$

As $\psi(x) < r$, we have $x \notin V_r$ and as $\theta(x) > s$ we have $x \in \overline{V}_s$ which again contradicts (1). Thus $\psi \ge \theta$, combining these inequalities gives $\psi = \theta$ and establishes the continuity of ψ.

Hence the result.

Regularity of Measure

Let μ be a measure defined on a σ-algebra M of subsets of X where X is a locally compact Hausdorff space. and suppose that M contains the Baire sets. A set $E \in M$ is said to be outer regular for μ(or μ is outer regular for E) if

$$\mu E = \text{Inf}\ \{\mu O : E \subset O, \text{ open}, O \in M\}$$

It is said to be inner regular if

$$\mu E = \sup\ \{\mu K : K \subset E, K \text{ compact}, K \in M\}$$

The set E is said to be regular for μ if it is both inner and outer regular for μ.

We say that the measure μ is inner regular (outer regular, regular) if it is inner regular (outer regular, regular) for each set $E \in M$. Lebesgue measure is a regular measure.

For compact spaces X, there is complete symmetry between inner regularity and outer regularity. A measurable set E is outer regular if and only if its complement is inner regular. A finite measure on X is inner regular if and only if it is outer regular, and hence regular. When X is compact, every Baire measure is regular.

Remark :- **When X is no longer compact, we lose this symmetry because the complement of an open set need not be compact.**

Proposition :- **Let μ be a finite measure defined on a σ-algebra M which contains all the Baire sets of a locally compact space X. If μ is inner regular, it is regular.**

Proof :- **Let $E \in M$, then**

$$\mu \overline{E} = \sup\ \{\mu K;\ K \subset \overline{E},\ K \in M \text{ and } K \text{ compact}\}.$$

But for each such a K, we have $\overline{K}$ open and $E \subset \overline{K}$. Hence

$$\begin{aligned} \mu E = \mu X - \mu \overline{E} &= \text{Inf}\{\mu X - \mu K\} \\ &= \text{Inf}\ \mu\ \overline{K} \\ &\geq \text{Inf}\ \{\mu O;\ E \subset O\} \end{aligned}$$

Thus

$$\mu E = \text{Inf}\ \{\mu O : E \subset O;\ O \text{ open and } O \in M\}.$$

Theorem : - **Let μ be a Baire measure on a locally compact space X and E a σ-bounded Baire set in X. Then for $\in > 0$,**

(iii) There is a σ - compact open set O with $E \subset O$ and $\mu\ (O \sim E) < \in$.

(iv) $\mu\ E = \sup\ \{\mu K\ ;\ K \subset E$, K a compact $G_\delta\}$.

Proof :- **Let R be the class of sets E that satisfy (i) and (ii) for each $\in > 0$. Suppose $E = UE_n$, where $E_n \in R$. Then for each n, there is a σ-compact open set O_n with $E_n \subset O_n$ and**

$$\mu (O_n \sim E_n) < 2^{-n} \in. \text{ Then } O = U\, O_n$$

is again a σ-compact open set with

$$\mu (O \sim E) \subset U (O_n \sim E_n)$$

and so

$$\mu(O \sim E) \leq \sum \mu(O_n \sim E_n) < \in$$

Thus E satisfies (i)

If for some, n, we have $\mu E_n = \infty$, then there are compact G_δ′s of arbitrary large finite measure contained in $E_n \subset E$. Hence (ii) holds for E. If $u E_n < \infty$ for each n, there is a $K_n \subset E_n$, K_n, a compact G_δ and

$$\mu(E_n \sim K_n) < 2^{-n}\in$$

Then

$$\mu E = \sup_N \mu\left(\bigcup_{n=1}^{N} E_n\right)$$

$$\leq \sup_N \mu\left(\bigcup_{n=1}^{N} K_n\right) + \in$$

Thus E satisfies (ii). If E is a compact G_δ, then there is a continuous real valued function φ with compact support such that $0 \leq \phi \leq 1$ and $E = \{x; \phi(x) = 1\}$. Let $O_n = \{x ; \phi(x) > 1-1/n\}$. Then O_n is a σ-compact open set with $\overline{O}_n$ compact. Since $\mu O_1 < \infty$, we have $\mu E = \text{Inf } \mu\, O_n$. Thus each compact G_δ satisfies (i) and it trivially satisfies (ii)

Let X be compact. Then E satisfies (i) if and only if $\overline{E}$ satisfies (ii) and so the collection R of sets satisfying (i) and (ii) is a σ-algebra containing the compact G_δ′s. Thus R contains all Baire sets and the prop. holds when X is compact.

For an arbitrary locally compact space X and bounded Baire set E, we can take H to be a compact G_δ and U to be a σ-compact open subsets of X such that

$$\overline{E} \subset U \subset H.$$

Then E is a Baire subset of H and so

$$\mu (W - E) < \in$$

Since W and U are σ-compact, so is $O = W \cap U$. Thus O is a σ-compact open set with $E \subset O \subset W$.

$$O \sim E \subset W \sim E$$

and

$$\mu(O \sim E) < \in.$$

Thus E satisfies (i). Therefore all bounded Baire sets are in R.

Since R is closed under countable unions and each σ-bounded Baire set is a countable union of bounded Baire sets, we see that every σ-bounded Baire set belongs to R .

Remark :- **If we had defined the class of Baire sets to be the smallest σ-ring containing the compact G_δ′s and taken a Baire measure to be defined on this σ-ring, then the above theorem takes the form**

"Every Baire measure is regular". If X is σ-compact, the σ-ring and the σ-algebra generated by the compact G_δ′s are the same. Hence we have the following corollary.

Corollary :- **If X is σ-compact, then every Baire measure on X is regular.**

Quasi-Measure :- **A measure μ defined on σ-algebra M which contains the Baire sets is said to be quasi-regular if it is outer regular and each open set $O \in M$ is inner regular for μ.**

A Baire measure on a space which is not σ-compact need not be regular but we can require it to be inner regular or quasi-regular without changing its values on the σ-bounded Baire sets.

Proposition :- **Let μ be a measure defined on a σ-algebra M containing the Baire sets. Assume either that μ is quasi-regular or that μ is inner regular. Then for each $E \in M$ with $\mu E < \infty$, there is a Baire set B with**

$$\mu(E \Delta B) = O$$

Proof :- **We consider only the quasi-regular case. Let E be a measurable set of finite measure. Since μ is outer regular, we can find a sequence $<O_n>$ of open sets with**

$$O_n \supset O_{n+1} \supset E$$

and

$$\mu O_n < \mu E + 2^{-n}$$

Since μ is quasi-regular, there is a compact set $K_m \subset O_m$ with

$$\mu K_m > \mu O_m - 2^{-m}$$

and we may take K_m to be a G_δ set by Lemma 1. Now

$$\mu K_m > \mu O_m - 2^{-m} \geq \mu E - 2^{-m}$$

$$> \mu O_n - 2^{-n} - 2^{-m}$$

Set

$$\mathbf{H_m} = \bigcup_{j=m}^{\infty} K_j$$

Then H_m is a Baire set, $H_m \subset O_m \subset O_n$ for $m \geq n$. Also $H_m \supset H_{m+1}$, and

$$\mu H_m \geq \mu K_m > \mu O_n - 2^{-n} - 2^{-m}.$$

Let $B = \cap H_m$. Then B is a Baire set, $B \subset O_n$ and

$$\mu B = \lim \mu H_m$$

Thus

$$\mu B \geq \mu O_n - 2^{-n},$$

Since $B \subset O_n$ and $E \subset O_n$, we have

$$B \,\Delta\, E \subset (O_n \sim B) \cup (O_n \sim E)$$

and so

$$\mu(B \,\Delta\, E) \leq \mu(O_n \sim B) + \mu(O_n \sim E)$$
$$< 2^{-n} + 2^{-n} = 2^{-n+1}$$

This is true for any n and so

$$\mu(B \,\Delta\, E) = 0.$$

Proposition :- **Let $\bar{\mu}$ be a non-negative extended real valued function defined on the class of open subsets of X and satisfying**

(vi) $\bar{\mu} O < \infty$ **if** $\bar{O}$ **compact.**

(vii) $\bar{\mu} O_1 \leq \bar{\mu} O_2$ **if** $O_1 \subset O_2$.

(viii) $\bar{\mu} (O_1 \cup O_2) = \bar{\mu} O_1 + \bar{\mu} O_2$ **if** $O_1 \cap O_2 = \phi$.

(ix) $\bar{\mu} (\cup O_i) \leq \sum \bar{\mu} O_i$

(x) $\bar{\mu} (O) == \sup \{\mu U;\ \bar{U} \subset O,\ \bar{U}$ **compact}**

Then the set function μ^* defined by

$$\mu^* E = \text{Inf}\ \{\bar{\mu} O;\ E \subset O\}$$

is a topologically regular outer measure.

Proof :- **The monotonicity and countable subadditivity of μ^* follow directly from (ii) and (iv) and the definition of μ^*. Also $\mu^* O = \bar{\mu} O$ for O open and so condition (ii) of the definition of regularity follows from hypothesis (iii) of the proposition and the condition (i) from the definition of μ^*. Since $\bar{\mu} O < \infty$ for $\bar{O}$ compact, we have $\mu^* E < \infty$ for each bounded set E.**

Riesz-Markov Theorem

Let X be a locally compact Hausdorff space. By $C_c(X)$, we denote as usual, the space of continuous real valued functions with compact support. A real valued linear functional I on $C_c(X)$ is said to be positive if $I(f) \geq 0$ whenever $f \geq 0$. The purpose of the following theorem is to prove that every positive linear functional on $C_c(X)$ is represented by integration with respect to a suitable Borel (or Baire) measure. In particular we have the following theorem :

Statement of Riesz-Markov Theorem

Let X be a locally compact Hausdorff space and I a positive linear functional on $C_c(X)$. Then there is a Borel measure μ on X such that

$$I(f) = \int f\, du$$

For each $f \in C_c(X)$. The measure μ may be taken to be quasi-regular or to be inner regular. In each of these cases it is then unique.

Proof :- **For each open set O define $\bar{\mu}$ O by**

$$\bar{\mu}\, O = \sup \{I(f);\ f \in C_c(X),\ O \leq f \leq 1,\ \sup f \subset O\}$$

Then $\bar{\mu}$ is an extended real valued function defined on all open sets and is readily seen to be monotone, finite on bounded sets and to satisfy the regularity (v) of the above Proposition. To see that $\bar{\mu}$ is countably subadditive on open sets, let $O = UO_i$ and let f be any function in $C_c(X)$ with $O \leq f \leq 1$ and $\sup f \subset O$. Thus there are non-negative functions $\phi_1, \phi_2, \ldots, \phi_n$ in $C_c(X)$ with $\sup \phi_i \subset O_i$ and

$$\sum_{i=1}^{n} \phi_i = 1.$$

on sup f. Then $f = \sum \phi_i f$, $o \leq \phi_i f \leq 1$ and

$\sup (\phi_i f) \subset O_i$. Thus

$$If = \sum_{i=1}^{n} I(\phi_i f) \leq \sum_{i=1}^{n} \bar{\mu}\, O_i$$

$$\leq \sum_{i=1}^{\infty} \bar{\mu}\, O_i$$

Taking the sup over all such f gives

$$\bar{\mu}\, O \leq \sum_{i=1}^{\infty} \bar{\mu}\, O_i$$

and $\bar{\mu}$ is countably subadditive.

If $O = O_1 \cup O_2$ with $O_1 \cap O_2 = \phi$ and $f_i \in C_c(X)$, $0 \leq f_i \leq 1$ and $\sup f_i \subset O_i$, then the function $f = f_1 + f_2$ has $\sup f \subset O$ and $0 \leq f \leq 1$. Thus

$$I f_1 + I f_2 \leq \bar{\mu}\, O.$$

Since f_1 and f_2 can be chosen arbitrarily, subject to $0 \le f_i \le 1$ and sup $f_i \subset O_i$, we have

$$\bar{\mu} O_1 + \bar{\mu} O_2 \le \bar{\mu} O,$$

whence

$$\bar{\mu} O_1 + \bar{\mu} O_2 = \bar{\mu} O$$

Thus $\bar{\mu}$ satisfies the hypothesis of the above proposition so $\bar{\mu}$ extends to a quasi-regular Borel measure.

We next proceed to show that If = $\int$ f du for each $f \in C_c(X)$. Since f is the difference of two non-negative functions in $C_c(X)$, it is sufficient to consider $f \ge 0$. By linearity we may also take $f \le 1$.

Choose a bounded open set O with sup $f \subset O$. Set

$$O_k = \{x;\ n f(x) > k-1\}$$

and $O_o = O$. Then $O_{n+1} = \phi$ and $\bar{O}_{k+1} \subset Ok$.

Define
$$\phi_k = \begin{cases} 1 & \text{in} \quad O_{k+1} \\ nf(x) - k + 1 & \text{in} \quad O_k - O_{k+1} \\ 0 & \text{in} \quad \bar{O}_k \end{cases}$$

Then $$f = \frac{1}{n}\sum_{k=1}^{n} \phi_k$$

We also have sup $\phi_k \subset \bar{O}_k \subset O_{k-1}$ and

$\phi_K = 1$ on O_{k+1}. Thus

$$\bar{\mu} O_{k+1} \le I\phi_k \le \bar{\mu} O_{k-1}$$

for $k \ge 1$.

Also

$$\bar{\mu} O_{k+1} \le \int \phi_k \, d\bar{\mu} \le \bar{\mu} O_k$$

for $k \ge 1$.

Hence

$$-\mu O_1 \le \sum_{k=1}^{n} (I\phi_k - \int \phi_k) \le \bar{\mu} O_0 + \bar{\mu} O_1$$

Consequently

$$|If - \int f du| \le \frac{2}{n} \bar{\mu} O$$

since n is arbitrary,

If = $\int$f d$\bar{\mu}$.

Thus there is an inner regular Borel measure μ which agrees with $\bar{\mu}$ on the σ-bounded Borel sets. Since only the values of μ on σ-bounded Baire sets enter into $\int$f du, we have

If = $\int$ f du.

The unicity of $\bar{\mu}$ and μ is obvious.

Unit-II

Normed Linear Spaces

First of all we introduce some sort of distance measuring device to vector spaces and ultimately introduce limiting notions. In other words, our aim is to study a class of spaces which are endowed with both a topological and algebraic structure. This combination of topological and algebraic structures opens up the possibility of studying linear transformations of one such space into another. First of all we give some basic concepts and definitions.

Definition 1. A vector space or linear vector space X is an additive Abelian group (whose elements are called vectors) with the property that any scalar α and any vector x can be combined by an operation called scalar multiplication to yield a vector αx in such a way that

(i) $\alpha(x + y) = \alpha x + \alpha y$

(ii) $(\alpha+\beta)x = \alpha x + \beta x$,

(iii) $(\alpha\beta)x = \alpha(\beta x)$

(iv) $1.x = x$

$\forall$ x, y $\in$ X and α, β are scalars. The two primary operations in a linear space–addition and scalar multiplication are called the linear operations. The zero element of a linear space is usually referred to as the origin.

A linear space is called a real linear space or a complex linear space according as the scalars are real numbers or complex numbers.

Definition 2. An isomorphism f between linear spaces (over the same scalar field) is a bijective linear map that is f is bijective and

$$f(\alpha x + \beta y) = \alpha f(x) + \beta\, f(y)$$

Two linear spaces are called isomorphic (or linearly isomorphic) if and only if there exists an isomorphism between them.

Definition 3. A semi-norm on a linear space X is a function $\rho : X \to R$ satisfying

(i) $\rho(x) \geq 0 \ \forall\ x \in X$.

(ii) $\rho(\alpha x) = |\alpha|\, \rho(x)$ for all $x \in X$ and α(scalar)

(iii) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$.

Property (i) is called absolute homogeneity of ρ and property (ii) is called subadditivity of ρ. Thus a semi-norm is non-negative real, subadditive, absolutely homogeneous function of the linear space e.g. $\rho(x) = |x|$ is a semi-norm on the linear space C of complex numbers. Similarly if $f : X \to C$ is a linear map, then $\rho(x) = |f(x)|$ is a semi-norm on X.

Thus a semi-normed linear space is an ordered pair (X, ρ) where ρ is a semi-norm on X.

Definition 4. A norm on a linear space X is a function $\|\ \| : X \to R$ satisfying

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ for $x \in X$

(ii) $\|\alpha x\| = |\alpha| . \|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

we observe that a semi-norm becomes a norm if it satisfies one additional condition i.e.

$$\|x\| = 0 \text{ iff } x = 0$$

Further, $\|x\|$ is called norm of x. The non-negative real number $\|x\|$ is considered as the length of the vector x.

A normed linear space is an ordered pair $(X, \|\cdot\|)$ where $\|\cdot\|$ is a norm on X.

Metric on Normed linear Spaces

Definition 5. Let X be an arbitrary set. It is called a metric space if there exists a function $d : X \times X \to R$ (called distance or metric function) satisfying

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ [Triangle inequality]

for any $x, y, z \in X$

(X, d) is called a metric space.

Let N be a normed linear space. We introduce a metric in N defined by

$$d(x, y) = || x-y||$$

This metric (distance function) satisfies all axioms of the definition of norm. Hence a normed linear space N is a metric space with respect to the metric d defined above. But every metric space need not be a normed linear space since in every metric space there need not be a vector space structure defined e.g. the vector space $X \neq 0$ with the discrete metric defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is not a normed linear space.

Remark :- In the definition of norm $||x|| = 0 \Leftrightarrow x = 0$ is equivalent to the condition

$$|x|| \neq 0 \text{ if } x \neq 0$$

Also the fact that $||x|| > 0$ is implied by the second and third condition of norm

$$||0|| = ||0.1|| = 0.||1|| = 0$$

and $||0|| = ||x-x|| \leq ||x|| + ||x|| = 2||x||$

$\Rightarrow$ $2||x|| \geq 0$

$\Rightarrow$ $||x|| \geq 0$.

Remark :- As in the case of real line, the continuity of a function can be given in terms of convergence of certain sequence. We can alternatively define continuity in terms of convergence of sequence in normed linear space also.

Definition 6. Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two normed linear spaces respectively. We say that f is continuous at $x_0 \in E$ if given $\in > 0$, $\exists\ \delta > 0$ whenever $\|x - x_0\|_E < \delta$

$$\Rightarrow \quad \|f(x) - f(x_0)\|_F < \in$$

Since every normed linear space is a metric space, this definition of continuity is same in it as the definition of continuity in metric space.

Thus f is continuous at $x_0 \in E$ iff

whenever $\quad x_n \to x_0$ in E

$\quad f(x_n) \to f(x_0)$ in F.

Remark : In normed linear spaces, convergence is defined as

$$x = \lim_n x_n \text{ or } x_n \to x \text{ by } \|x_n - x\| \to 0 \text{ as } n \to \infty$$

This convergence in normed linear space is called **convergence in norm or strong convergence.**

Definition 7. A sequence $<x_n>$ in a normed linear space is a Cauchy sequence if given $\in > 0$, there exists a positive integer m_0 such that

$$m, n \geq m_0 \Rightarrow \|x_m - x_n\| < \in.$$

Definition 8. A normed linear space N is called complete or Banach space iff every Cauchy sequence in it is convergent that is if for each Cauchy sequence $<x_n>$ in N, there exist an element x_0 in N such that $x_n \to x_0$. A complete normed linear space is called a Banach space.

Some properties of Normed Linear Spaces

Theorem 1. Let N be a normed linear space over the scalar field F. Then

(i) The map $(\alpha, x) \to \alpha x$ from $F \times N \to N$ is continuous

(ii) The map $(x, y) \to x+y$ from $N \times N \to N$ is continuous.

(iii) The map $x \to \|x\|$ from N to R is continuous.

Proof :- To prove (i) we must show that if $\alpha_n \to \alpha$ and $x_n \to x$, then $\alpha_n x_n \to \alpha x$. So we assume $\alpha_n \to \alpha$ and $x_n \to x$ i.e. $|\alpha_n - \alpha| \to 0$, $\|x_n - x\| \to 0$.

Then $\|\alpha_n x_n - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\|$

$$\leq |\alpha_n|\ \|x_n - x\| + |\alpha_n - \alpha|.\ \ \|x_n\| \to 0$$

and so (i) holds.

To prove (ii) we suppose that $x_n \to x$, $y_n \to y$ i.e. $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$/

Then by triangle inequality

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\|$$

$$\leq \|x_n - x\| + \|y_n - y\| \to 0$$

and so $x_n + y_n \to x + y$ and hence (ii) holds. Before proving (iii), we establish the inequality

$$|\ \|x\| - \|y\|\ | \leq \|x - y\| \qquad \ldots(*)$$

We note that in a normed linear space

$$\|x\| = \|y + (x - y)\| \leq \|y\| + \|x - y\|$$

$$\Rightarrow \qquad \|x\| - \|y\| \leq \|x - y\| \qquad \ldots(1)$$

On interchanging the roles of x and y, we find that

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\| \qquad \ldots(2)$$

From (1) and (2), it follows that

$$|\ \|x\| - \|y\|\ | \leq \|x - y\|$$

We now prove (iii). Let $x_n \to x$, then from the above inequality,

$$|\ \|x_n\| - \|x\|\ | \leq \|x_n - x\| \to 0$$

which implies that $\|x_n\| \to \|x\|$ Thus we have shown that $x_n \to x \Rightarrow \|x_n\| \to \|x\|$.

Thus the map $\|\cdot\| : N \to R$ is continuous. Hence the result.

Remark (1) (i) and (ii) show that scalar multiplication and addition are jointly continuous where as (iii) shows that norm is a continuous function.

(2) The introduction of a norm in a linear space is called norming

Theorem 2 :- In a normed linear space, every convergent sequence is a Cauchy sequence.

Proof : Suppose that the sequence $<x_n>$ in a normed linear space N converges to a point $x_0 \in N$. To show that it is Cauchy sequence, let $\in > 0$ be given. Since the sequence $<x_n>$ converges to x_0, there exists a positive integer m_0 such that $n \geq m_0 \Rightarrow \|x_n - x_0\| < \frac{\in}{2}$. Hence for all m, $n \geq m_0$, we have

$$\|x_m - x_n\| = \|x_m - x_0 + x_0 - x_n\| \leq \|x_m - x_0\| + \|x_n - x_0\| < \frac{\in}{2} + \frac{\in}{2} = \in.$$

Thus the convergent sequence $<x_n>$ is a Cauchy sequence.

Further Properties of Normed spaces

By definition , a subspace Y of a normed space X is a subspace of X considered as a vector space , with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be induced by the norm on X. If Y is closed in X , then Y is called a closed subspace of X. Thus , a subspace Y of a Banach X is considered as a normed space. Hence we donot require Y to be complete.

Theorem 1: A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Proof : The result directly follows from "A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Definition :- Infinite series can now be defined in a way similar to that in calculus. In fact, if $< x_k >$is a sequence in a normed space X , we can associate with $< x_k >$ the sequence $<S_n >$ of partial sums

$$S_n = x_1 + x_2 + \dots\dots\dots + x_n$$

For n = 1, 2,…… If $< S_n >$ is convergent , say $S_n \to S$ that is $\| S_n - S \| \to 0$,

Then the infinite series or briefly the series

$$\sum_{K=1}^{\infty} x_K = x_1 + x_2 + \dots\dots\dots \quad (1)$$

is said to converge or to be convergent , S is called the sum of the series and we write

$$S = \sum_{K=1}^{\infty} x_K = x_1 + x_2 + \dots\dots\dots$$

If $\| x_1 \| + \| x_2 \| + \ldots\ldots..$ converges ,then the series (1) is said to be absolutely convergent. However in a normed space X absolute convergence implies convergence if and only if X is complete.

The concept of convergence of a series can be used to define a basis as follows :

If a normed space x contains a sequence $< e_n >$ with the property that for every $x \in X$, there is a unique sequence of scalars $< \alpha_n >$ such that

$$\| x - (\alpha_1 e_1 + \ldots\ldots\ldots.. + \alpha_n e_n) \| \to 0 \qquad \text{as } n \to \infty \qquad (6)$$

then $< e_n >$ is called a Schauder Basis for X. The series

$$\sum_{K=1}^{\infty} \alpha_K e_K$$

which has the sum x is then called the expansion of x with respect to $< e_n >$ and we write

$$x = \sum_{K=1}^{\infty} \alpha_K \; e_K$$

Finite Dimensional Normed Spaces and Subspaces

Theorem : Every finite dimensional subspace Y of a normed space X is complete. In particular , every finite dimensional normed space is complete.

To prove the theorem , we prove a Lemma ,

Lemma : Let $\{x_1 , x_2 ,\ldots\ldots\ldots., x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number $C > 0$ such that for every choice of scalars $\alpha_1 , \alpha_2 , \ldots\ldots., \alpha_n$, we have

$$\| \alpha_1 x_1 + \ldots\ldots\ldots. + \alpha_n x_n \| \geq C (| \alpha_1 | + \ldots\ldots. + | \alpha_n |) \qquad (C > 0) \quad (1)$$

Proof : We write $S = | \alpha_1 | + | \alpha_2 | + \ldots\ldots + | \alpha_n |$. If $S = 0$, all α_i are zero , so that (1) holds for any C. Let $S > 0$, then (1) is equivalent to the inequality which we obtain from (1) by dividing by S and writing $\beta_j = \alpha_j/S$ that is

$$\| \beta_1 x_1 + \ldots\ldots\ldots + \beta_n x_n \| \geq C \qquad \left(\sum_{j=1}^{n} | \beta_j | = 1 \right) \qquad (2)$$

Hence it is sufficient to prove the existence of a $C > 0$ such that (2) holds for every n-tuple of scalars $\beta_1 ,\ldots\ldots. , \beta_n$ with

$$\Sigma \, | \beta_j | = 1 \, .$$

Suppose that this is false. Then there exists a sequence $< y_m >$ of vectors

$$y_m = \beta_1^{(m)} x_1 + \ldots\ldots\ldots + \beta_n^{(m)} x_n \qquad \left(\sum_{j=1}^{n} | \beta_j^{(m)} | = 1 \right)$$

such that

$$\| y_m \| \to 0 \qquad \text{as } m \to \infty.$$

Since $\Sigma \mid \beta_j^{(m)} \mid = 1$, we have $\mid \beta_j^{(m)} \mid \leq 1$. Hence for each fixed j , the sequence

$$(\beta_j^{(m)}) = (\beta_j^{(1)} , \beta_j^{(2)} ,\ldots\ldots.)$$

is bounded. Consequently , by the Bolzano - Weierstrass theorem , $(\beta_1^{(m)})$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $< y_{1,m} >$ denote the corresponding subsequence of $< y_m >$. By the same argument , $<y_{1,m} >$ has a subsequence $< y_{2,m} >$ for which the corresponding subsequence of scalars $\beta_2^{(m)}$ converges , let β_2 denote the limit –continuing in this way , after n steps we obtain a subsequence

$$< y_{n,m} > = (y_{n,1} , y_{n,2} ,\ldots\ldots\ldots.) \qquad \text{of } < y_m >$$

whose terms are of the form

$$y_{n,m} = \sum_{j=1}^{n} \gamma_j^{(m)} x_j \qquad \left(\sum_{j=1}^{n} | \gamma_j^{(m)} | = 1 \right)$$

with scalars $\gamma_j^{(m)}$ satisfying $\gamma_j^{(m)} \to \beta_j$ as $m \to \infty$.

Hence as $m \to \infty$,

$$y_{n,m} \to y = \sum_{j=1}^{n} \beta_j \; x_j$$

where $\Sigma \ |\beta_j| = 1$ so that not all β_j can be zero. Since $\{x_1 ,\ldots\ldots , x_n \}$ is a linearly independent set , we thus have $y \neq 0$. On the other hand , $y_{n,m} \to y$ implies $\| y_{n,m} \| \to \| y \|$ by the continuity of the norm. Since $\| y_m \| \to 0$ by assumption and $< y_{n,m} >$ is a subsequence of $< y_m >$, we must have $\| y_{n,m} \| \to 0$. Hence $\| y \| = 0$, so that $y = 0$. But this contradicts that $y \neq 0$, and the lemma is proved.

Now we prove the theorem.

Proof of the theorem : We consider an arbitrary Cauchy sequence $< y_m >$ in Y and show that it is convergent in Y, the limit will be denoted by y. Let dim Y =

n and $\{e_1, e_2, \ldots\ldots, e_n\}$ any basis for Y. Then each y_m has a unique representation of the form

$$y_m = \alpha_1^{(m)} e_1 + \ldots\ldots\ldots + \alpha_n^{(m)} e_n$$

Since $< y_m >$ is a Cauchy sequence , for every $\in > 0$, there is an N such that $\| y_m - y_n \| < \in$ when m , r > N. From this and the above Lemma , we have for some C > 0 ,

$$\in > \| y_m - y_r \| = \| \sum_{j=1}^{r} (\alpha_j^{(m)} - \alpha_j^{(r)}) e_j \|$$

$$\geq C \sum_{j=1}^{r} | \alpha_j^{(m)} - \alpha_j^{(r)} |$$

where m , r > N. Division by C > 0 gives

$$| \alpha_j^{(m)} - \alpha_j^{(r)} | \leq \sum_{j=1}^{n} | \alpha_j^{(m)} - \alpha_j^{(r)} | < \frac{\in}{C} \qquad (m , r > N)$$

This shows that each of the n sequences

$$(\alpha_j^{(m)}) = (\alpha_j^{(1)} , \alpha_j^{(2)} , \ldots\ldots\ldots) \qquad j = 1, 2 , \ldots, n.$$

is Cauchy in R or C. Hence it converges let α_j denote the limit. Using these n limits , $\alpha_1 , \alpha_2 , \ldots , \alpha_n$, we define

$$y = \alpha_1 e_1 + \alpha_2 e_2 + \ldots\ldots\ldots + \alpha_n e_n$$

clearly $y \in Y$. Further

$$\| y_m - y \| = \| \sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j) e_j \| \leq \sum_{j=1}^{n} | \alpha_j^{(m)} - \alpha_j \| . \| e_j \|$$

On the right $\alpha_j^{(m)} \to \alpha_j$. Hence $\| y_m - y \| \to 0$, that is $y_m \to y$. This shows that $< y_m >$ is convergent in Y. Since $<y_m>$ was an arbitrary Cauchy sequence in Y, This proves that Y is complete.

Remark : From the above theorem and the result "A subspace M of a complete metric space X is complete if and only if the set M is closed in X" , we get the following :

Theorem : Every finite dimensional subspace Y of a normed space X is closed in X.

Remark : Infinite dimensional subspaces need not be closed e.g. Let X = C[0, 1] and Y = span$\{x_0 , x_1 , \ldots\ldots\}$ where $x_j(t) = t^j$ so that Y is the set of polynomials. Y is not closed in X.

Quotient Space

Definition 9. Let M be a subspace of a linear space L and let the coset of an element x in L be defined by

$$x + M = \{ x + m ; m \in M\}$$

Then the distinct cosets form a partition of L and if addition and scalar multiplication are defined by

$$(x + M) + (y+M) = (x+y) + M$$

and $$\alpha(x + M) \equiv \alpha x + M$$

then these cosets constitute a linear space denoted by L/M and called the quotient space of L with respect to M. The origin in L/M is the coset 0 +M = M and the negative of

$$x + M \text{ is } (-x) + M$$

Theorem 3. Let M be a closed linear subspace of a normed linear space N. If the norm of a coset x +M in the quotient space N/M is defined by

$$\| x+M\| = \text{Inf } \{\|x+m\|; m \in M\} \qquad \ldots(1)$$

Then N/M is a normed linear space. Further if N is a Banach space. Then so is N/M.

Proof :- We first verify that (1) defines a norm in the required sense. It is obvious that $\|x+M\| \geq 0$. since $\| x+m\|$ is a non-negative real number and every set of non-negative real numbers is bounded below, it follows that inf $\{\|x+m\|; m\in M\}$ is non negative. That is

$$\|x + M\| \geq 0 \; \forall \; x + M \in N/M$$

Also $\|x + M\| = 0 \Leftrightarrow$ there exists a sequence $\{m_k\}$ in M such that $\|x + m_k\| \to 0$

$\Leftrightarrow$ x is in M

$\Leftrightarrow$ x + M = M = The zero element of N/M.

Next we have

$$\|(x + M) + (y + M) = \|(x + y) + M\|$$

$$= \text{Inf } \{\|x + y + m\|; m \in M\}$$

$$= \text{Inf } \{\|x + y + m + m'\|; m \text{ and } m' \in M\}$$

$$= \text{Inf } \{\|(x + m) + (y+m')\| ; m, m' \in M\}$$

$$\le \text{Inf}\ \{\|x + m\|;\ m \in M\} + \text{Inf}\ \{\|y + m'\|.\ m' \in M\}$$

$$= \|x + M\| + \|y + M\|$$

$$\|\ \alpha(x+M)\ \| = \text{Inf}\ \{\|\alpha(x + m)\ \|;\ m \in M\}$$

$$= \text{Inf}\ \{|\alpha|\ \|x+m\|;\ m \in M\}$$

$$= |\alpha|\ \text{Inf}\ \{\|x+m\|;\ m \in M\}$$

$$= |\alpha|\ \|x+M\|$$

Finally we assume that N is complete and we show that N/M is also complete. If we start with a Cauchy sequence in N/M, Then it is sufficient to show that this sequence has a convergent subsequence. It is clearly possible to find a subsequence $\{x_n + M\}$ of the original Cauchy sequence such that

$$\|(x_1 + M) - (x_2 + M)\| < \frac{1}{2}$$

$$\|\ (x_2 + M) - (x_3 + M)\ \| < \frac{1}{4}$$

and in general

$$\|\ (x_n + M) - (x_{n+1} + M)\| < \frac{1}{2^n}$$

we prove that this sequence is convergent in N/M. We begin by choosing any vector y_1 in $x_1 + M$ and we select y_2 in $x_2 + M$ such that $\|y_1 - y_2\| < \frac{1}{2}$. We next select a vector y_3 in $x_3 + M$ such that $\|y_2 - y_3\| < \frac{1}{4}$. Continuing in this way we obtain a sequence $\{y_n\}$ in N such that $\|y_n - y_{n+1}\| < \frac{1}{2^n}$. If $m < n$, then

$$\|y_m - y_n\| = \|y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \ldots + (y_{n-1} - y_n)\|$$

$$\le \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \ldots + \|y_{n-1} - y_n\|$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \frac{1}{2^{n-1}}$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots +$$

$$= \frac{(1/2)^m}{1-\frac{1}{2}} = \frac{1}{2^{m+1}}$$

So $\{y_n\}$ is a Cauchy sequence in N. Since N is complete, there exists a vector y in N such that $y_n \to y$. Finally

$$\|(x_n+M) - (y + M)\| = \|x_n - y + M\|$$

$$\leq \text{Inf } \{\|x_n - y + m\|;\ m \in M\}$$

$$\leq \|x_n + m - y\| \text{ for all } m \in M$$

But $y_n = x_n + m_n$ for some $m_n \in M$

$$\leq \|y_n - y\| \to 0 \text{ since } y_n \to y.$$

Hence $x_n + M \to y + M \in N/M$

$\Rightarrow$ N/M is complete.

Definition 10. A series $\sum_{n=1}^{\infty} a_n$, $a_n \in X$ is said to be convergent to $x \in X$, where X is a normed linear space if the sequence of partial sums $<S_n>$ where $S_n = \sum_{i=1}^{n} a_i$ converges to x i.e. for every $\in > 0$, there exists $n_0 \in N$ such that $\|S_n - x\| < \in$ for $n \geq n_0$. A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|a_n\|$ is convergent.

Since every normed linear space is a metric space, hence every convergent sequence in it is Cauchy but not conversely.

The following theorem gives a nice characterization of a Banach space in terms of series.

Theorem 4 :- A normed linear space is complete if and only if every absolutely convergent series in X is convergent.

Proof :- Let X be complete. For each positive integer n, let x_n be an element of X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Let $y_k = \sum_{n=1}^{K} x_n$. Then

$$\|y_{p+k} - y_k\| = \left\| \sum_{n=1}^{k+p} x_n - \sum_{n=1}^{k} x_n \right\|$$

$$= \left\| \sum_{n=k+1}^{k+p} x_n \right\|$$

$$\leq \sum_{n=k+1}^{k+p} \|x_n\| \to 0 \text{ as } k\to\infty.$$

Hence $< y_k >_{k=1}^{\infty}$ is a Cauchy sequence in X and since X is complete, there exists $x\in X$ such that

$$x = \lim_{k\to\infty} y_k = \lim_{k\to\infty} \sum_{n=1}^{k} x_n = \sum_{n=1}^{\infty} x_n$$

Thus the series $\sum_{n=1}^{\infty} x_n$ converges.

Conversely suppose every absolutely convergent series in X is convergent. Let $<x_n>$ be a Cauchy sequence in X. For each positive integer k, there is a positive integer n_k such that

$$\|x_n - x_m\| < \frac{1}{2^k} \text{ for all } n, m \geq n_k.$$

Choose $n_{k+1} > n_k$. Let $y_1 = x_{n_1}$ and

$$y_{k+1} = x_{n_{k+1}} - x_{n_k}, k \geq 1.$$

$\Rightarrow$ $\sum_{k=1}^{\infty} \|y_k\| < \infty$. Therefore there exists $y\in X$ such that

$$y = \lim_{m\to\infty} \sum_{k=1}^{m} y_k = \lim_{m\to\infty} x_{n_m}$$

Since $<x_n>$ is cauchy, $\lim_{n\to\infty} x_n$ is also y.

Hence the result.

Riesz Lemma :- Let X be a proper closed linear subspace of a normed linear space X over the field K. Let $0 < \alpha < 1$, then $\exists\, x_\alpha \in X$ such that

$$\|x_\alpha\| = 1 \text{ and } \inf_{y\in Y} \|x_\alpha - y\| \geq \alpha.$$

Theorem 5 :- Let X be normed linear space. The closed unit ball

$$B = \{x\in X ; \|x\| \leq 1\}$$

in X is compact if and only if X is finite dimensional.

Proof :- Let X be finite dimensional. Since B is closed and bounded. It follows from Heine-Borel theorem that it is compact.

Conversely suppose that B is compact but X is infinite dimensional. Choose $x_1 \in X$ with $\|x_1\| = 1$. This x_1 generates a one-dimensional subspace X_1 of X. Since every finite dimensional subspace of a normed linear space is closed, it follows that X_1 is closed. Now X_1 is a proper subspace of X and dim X = ∞. By Riesz-Lemma there is an $x_2 \in X$ of norm 1 such that

$$\|x_2 - x_1\| \geq \frac{1}{2}.$$

The set $\{x_1, x_2\}$ generates a two dimensional proper closed subspace X_2 of X. By Riesz Lemma, there is an x_3 of norm 1 such that for all $x \in X_2$, we have

$$\|x_3 - x\| \geq \frac{1}{2}$$

In particular

$$\|x_3 - x_1\| \geq \frac{1}{2}$$

and $$\|x_3 - x_2\| \geq \frac{1}{2}.$$

Proceeding by induction, we obtain a sequence $<x_n>$ of elements of B such that

$$\|x_m - x_n\| \geq \frac{1}{2} \quad (m \neq n)$$

i.e. $\{x_n\}$ can not have a convergent subsequence which contradicts the compactness of B. Hence the result.

Examples of Banach Spaces

The scalar field in each of the following examples will be either R or C whichever is appropriate.

Example 1: Consider linear spaces R and C of real numbers and complex numbers respectively. We introduce norm of a number x in R or C by defining $\| x \| = | x |$. Under this norm, both R and C are Banach spaces.

Example 2: Consider the linear spaces R^n and C^n of all n tuples x = (x_1, x_2,…..,x_n) of real and complex numbers. These spaces can be made into normed linear spaces by introducing the norm defined by $\| x \| = \left(\sum_{i=1}^{n} | x_i |^2 \right)^{\frac{1}{2}}$

We obtain n – dimensional Euclidean and unitary spaces both of which are complete and hence Banach. It can be easily verified that the norm introduced satisfies first two properties of norm. To show the validity of triangle inequality, we need the following two inequalities.

Cauchy's inequality: Let x = (x_1, x_2,…..,x_n) and y = (y_1, y_2,…..,y_n) be two n – tuples of real or complex numbers. Then

$$\sum_{i=1}^{n} | x_i \, y_i | \le \left(\sum_{i=1}^{n} | x_i |^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} | y_i |^2 \right)^{\frac{1}{2}}$$

Proof: We first remark that if a and b are any two non – negative real numbers, then $a^{1/2}. b^{1/2} \le \frac{a+b}{2}$. Infact, on squaring both sides and rearranging, it is equivalent to $0 \le (a - b)^2$ which is obviously true. If x = 0 or y = 0, the assertion is clear. We therefore assume that x ≠ 0 or y ≠ 0. We define a_i and b_i by

$$a_i = \left(\frac{| x_i |}{\| x \|} \right)^2 \quad \text{and} \quad b_i = \left(\frac{| y_i |}{\| y \|} \right)^2 .$$

Since $a^{1/2}. b^{1/2} \le \frac{a+b}{2}$.

$$\Rightarrow \quad \frac{| x_i \, y_i |}{\| x \| . \| y \|} \le \frac{| x_i |^2 / \| x \|^2 + | y_i |^2 / \| y \|^2}{2}$$

Summing these inequalities as i varies from 1 to n, we obtain

$$\frac{\sum_{i=1}^{n} | x_i \, y_i |}{\| x \| . \| y \|} \le \frac{1+1}{2} = 1$$

and hence

$$\sum_{i=1}^{n} | x_i \, y_i | \le \| x \| . \| y \|$$

which proves $\sum_{i=1}^{n} | x_i \, y_i | \le \| x \| . \| y \|$.

Minkowski's – inequality : Let x = $(x_1, x_2, \ldots\ldots, x_n)$ and y = $(y_1, y_2, \ldots\ldots., y_n)$ be two n – tuples of real or complex numbers. Then

$$\left(\sum_{i=1}^{n} | x_i + y_i |^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} | x_i |^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} | y_i |^2\right)^{\frac{1}{2}} .$$

or $$\| x + y \| \le \| x \| + \| y \|$$

Proof: Using Cauchy's inequality, we have the following chain of relations.

$$\| x + y \|^2 = \sum_{i=1}^{n} | x_i + y_i | . | x_i + y_i |$$

$$\le \sum_{i=1}^{n} | x_i + y_i | (| x_i | + | y_i |)$$

$$= \sum_{i=1}^{n} | x_i + y_i | . | x_i | + \sum_{i=1}^{n} | x_i + y_i | . | y_i |$$

$$\le \| x + y \| . \| x \| + \| x + y \| . \| y \|$$

$$= \| x + y \| (\| x \| + \| y \|)$$

If $\| x + y \| = 0$, the inequality to be proved is trivially true. If $\| x + y \| \neq 0$, then dividing the inequality (1) through by $\| x + y \|$, we obtain

$$\| x + y \| \le \| x \| + \| y \| .$$

and Minkowski inequality is established.

It follows from Minkowski inequality that triangle inequality in $\mathfrak{R}^n$ or C^n holds. Hence R^n or C^n are normed linear spaces with respect to co-ordinate wise addition and scalar multiplication and the norm defined by $\| x \| = \left(\sum_{i=1}^{n} | x_i |^2\right)^{\frac{1}{2}}$

We further claim that R^n and C^n are complete and hence Banach. We prove the completeness of R^n. The proof for C^n is similar. Let $< f_m >$ be a Cauchy sequence in R^n. If $\in > 0$ is given, then for all sufficiently large m and m′, we have $\| f_m - f_{m'} \| < \in$, $\| f_m - f_{m'} \| < \in^2$ and $\sum_{i=1}^{n} | f_m(i) - f_{m'}(i) |^2 < \in^2$ and from this we observe that $| f_m(i) - f_{m'}(i) | < \in$ for each i and all sufficiently large m and m′. The sequence $< f_m >$ therefore converges pointwise to a limit function f defined by $f(i) = \lim f_m(i)$. Since the set {1, 2, ….,n} is finite.

This convergence is uniform. We can thus find a positive integer m_0 such that $| f_m(i) - f(i) | < \frac{\in}{n^{1/2}}$ for all $m \geq m_0$ and every i. Squaring each of these inequalities and summing as i varies from 1 to n yields

$$\sum_{i=1}^{n} | f_m(i) - f(i) |^2 < \in^2 \text{ or } \| f_m - f \| < \in \text{ for all } m \geq m_0 .$$

This shows that the Cauchy sequence $< f_m >$ converges to the limit f and so R^n is complete.

Example 3: Let p be a real number such that $1 \leq p < \infty$. We denote by l_p^n, the space of all n – tuples $x = (x_1, x_2, \ldots, x_n)$ of scalars with the norm defined by

$$\| x \|_p = \left(\sum_{i=1}^{n} | x_i |^p \right)^{\frac{1}{p}}$$

Since the norm defined in the last example is obviously the special case of this norm which corresponds to p = 2, so the real and complex spaces l_2^n are the n – dimensional Euclidean and unitary spaces R^n and C^n. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ and let α be any scalar. Then l_p^n is a linear spaces with respect the operations

$$x + y = (x_1 + y_1, \ldots, x_n + y_n) \text{ and}$$

$$\alpha x = (\alpha x_1, \ldots, \alpha x_n)$$

Since the norm introduced above is non – negative and absolute homogeneous, so to show that l_p^n is a normed linear space, it is sufficient to prove that

$$\| x + y \|_p \leq \| x \|_p + \| y \|_p .$$

To show this, we first establish the following inequalities.

Holder's inequality: Let p and q be real numbers greater than 1, with the properties that $\frac{1}{p} + \frac{1}{q} = 1$ (Such numbers are called conjugate indices). Then for any complex number

$$x = (x_1, x_2, \ldots, x_n) \text{ and } y = (y_1, y_2, \ldots, y_n).$$

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

or in our notations

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \,.\, \|y\|_q$$

Proof: If x = 0 or y = 0, the inequality is obvious. So assume that both are non – zero. Set

$$a_i = \left(\frac{x_i}{\|x\|_p}\right)^p \quad \text{and} \quad b_i = \left(\frac{y_i}{\|y\|_q}\right)^q$$

Then using

$$a_i^{1/p}\, b_i^{1/q} \leq \frac{a_i}{p} + \frac{b_i}{q} \quad (a, b \geq 0)$$

We have

$$\frac{|x_i\, y_i|}{\|x\|_p \|y\|_q} \leq \frac{a_i}{p} + \frac{b_i}{q}$$

or

$$\frac{|x_i\, y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p}\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|y\|_q^p}$$

Summing these inequalities as i varies from 1 to n, we have

$$\frac{\sum_{i=1}^{n} |x_i\, y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p}\frac{\sum_{i=1}^{n} |x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{\sum_{i=1}^{n} |y_i|^q}{\|y\|_q^p}$$

$$= \frac{1}{p}\frac{(\|x\|_p)^p}{\|x\|_p^p} + \frac{1}{q}\frac{(\|y\|_q)^q}{\|y\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \,.\, \|y\|_q$$

We notice that when p = q = 2. Holder’s inequality converts into Cauchy’s inequality.

Minkowski’s inequality: Let p be a real number such that $p \geq 1$. Then for any complex numbers

$$x = (x_1, \ldots, x_n) \text{ and } y = (y_1, y_2, \ldots, y_n)$$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

or $\qquad \| x + y \|_p \le \| x \|_p + \| y \|_p$

Proof: The inequality is trivial when p = 1. So assume p > 1. Using Holder's inequality, we obtain

$$\| x + y \|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p$$

$$= \sum_{i=1}^{n} |x_i + y_i| . |x_i + y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

$$\le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$+ \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}}$$

Since (p – 1)q = p, we have

$$= \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}.\frac{p}{q}}$$

$$+ \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}.\frac{p}{q}}$$

$$= \| x \|_p . \| x + y \|_p^{p/q} + \| y \|_p . \| x + y \|_p^{p/q}$$

$$= (\| x \|_p + \| y \|_p) . (\|x + y \|_p^{p/q})$$

If $\| x + y \|_p = 0$, then the result is trivial. If $\| x + y \|_p \ne 0$, then dividing inequality (1), throughout by $\| x + y \|_p^{p/q}$, we obtain

$$\frac{\| x + y \|_p^p}{(\| x + y \|_p^{p/q})} \le (\| x \|_p + \| y \|_p) . \left(\frac{\| x + y \|_p^{p/q}}{\| x + y \|_p^{p/q}}\right)$$

$$\Rightarrow \| x + y \|_p^{p - \frac{p}{q}} \le \| x \|_p + \| y \|_p$$

$$\Rightarrow \| x + y \|_p^{p\left(1 - \frac{1}{q}\right)} \le \| x \|_p + \| y \|_p$$

$$\Rightarrow \| x + y \|_p^1 \le \| x \|_p + \| y \|_p \text{ since } \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \frac{1}{p} = 1 - \frac{1}{q}$$

Thus $\| x + y \|_p \le \| x \|_p + \| y \|_p$.

In view of the Minkowski's inequality, it follows that l_p^n is a normed linear space.

Now we prove completeness of l_p^n.

Let $< x_m >_{m=1}^{\infty}$ be a Cauchy sequence in l_p^n.

We write

$$x_m = (x_1^m, x_2^m, \ldots\ldots, x_n^m)$$

Let $\in > 0$ be given, since $< x_m >$ is a Cauchy sequence, there exists a +ve integer m_0 such that

$$l, m \ge m_0 \Rightarrow \| x_m - x_l \|_p < \in$$

$$\Rightarrow \| x_m - x_l \|_p^p < \in^p$$

$$\Rightarrow \sum_{i=1}^{n} | x_i^{(m)} - x_i^{(l)} |^p < \in^p \qquad (1)$$

$$\Rightarrow | x_i^{(m)} - x_i^{(l)} |^p < \in^p \quad i = 1, 2, \ldots\ldots, n.$$

$$\Rightarrow | x_i^{(m)} - x_i^{(l)} | < \in$$

This shows that the sequence $< x_i^m >_{m=1}^{\infty}$ is a Cauchy sequence in C or R and completeness of R and C implies that each of these sequence converges to a point say z_i in C or R such that

$$\lim_{m \to \infty} x_i^{(m)} = z_i \qquad (i = 1, 2, \ldots, n) \qquad (2)$$

we will now show that the Cauchy sequence $< x_m >$ converges to the point $z = (z_1, z_2, \ldots\ldots, z_n) \in l_p^n$. To prove this let $i \to \infty$ in (1), they by (2) for $m \ge m_0$, we have

$$\sum_{i=1}^{n} | x_i^{(m)} - z_i |^p < \in^p \Rightarrow \| x_m - z \|_p^p < \in^p$$

$$\Rightarrow \| x_m - z \|_p < \in$$

Consequently the Cauchy sequence < x_m > converges to z $\in l_p^n$. Hence l_p^n is complete and therefore it is a Banach space.

Example 4: Let p be a real number such that $1 \le p \le \infty$ and l_p denote the space of all sequences x = <x_1, x_2,……,x_n,…….> of scalars s that $\sum_{n=1}^{\infty} | x_n |^p < \infty$.

Show that l_p is a Banach space under the norm

$$\| x \|_p = \left[\sum_{n=1}^{\infty} | x_n |^p \right]^{\frac{1}{p}}$$

Solution: [N_1]: Since each $\sum_{n=1}^{\infty} | x_n |^p \ge 0 \Rightarrow$ we have $\| x \|_p \ge 0$

and $\| x \|_p = 0$

$$\Leftrightarrow \left[\sum_{n=1}^{\infty} | x_n |^p \right]^{\frac{1}{p}} = 0 \Leftrightarrow \sum_{n=1}^{\infty} | x_n |^p = 0$$

$$\Leftrightarrow | x_n |^p = 0 \quad \forall\ n = 1, \ldots..,\infty$$

$$\Leftrightarrow x_n = 0 \quad \forall\ n = 1, \ldots\ldots,\infty$$

$$\Leftrightarrow x = < x_1, x_2,\ldots..,x_n\ldots.. > = 0$$

[N_2] is $\quad \| x + y \|_p \le \| x \|_p + \| y \|_p$

$$\Rightarrow \quad \| x + y \|_p = \left[\sum_{n=1}^{\infty} | x_n + y_n |^p \right]^{\frac{1}{p}} \quad (1 \le p \le \infty)$$

$$\le \left[\sum_{n=1}^{\infty} | x_n |^p \right]^{\frac{1}{p}} + \left[\sum_{n=1}^{\infty} | y_n |^p \right]^{\frac{1}{p}}$$

[Minkowski's inequality for sequence]

$$= \| x \|_p + \| y \|_p$$

[N_3]

$$\| \alpha x \|_p = \left[\sum_{n=1}^{\infty} | \alpha x_n |^p \right]^{\frac{1}{p}}$$

$$= \left[\sum_{n=1}^{\infty} | \alpha |^p | x_n |^p \right]^{\frac{1}{p}}$$

$$= | \alpha | \left[\sum_{n=1}^{\infty} | x_n |^p \right]^{\frac{1}{p}} = | \alpha | . \| x \|_p.$$

Thus l_p is a normed linear space.

To prove that l_p is complete.

Let $< x_n >_{n=1}^{\infty}$ be a Cauchy sequence in l_p. Since each x_n is itself a sequence of scalars. We shall denote an element x_m by

$$x_m = < x_1^{(m)}, x_2^{(m)}, \ldots\ldots\ldots, x_n^{(m)}, \ldots. >$$

Where $\sum_{n=1}^{\infty} | x_n^{(m)} |^p < \infty$. Since each $< x_n >$ is a cauchy sequence in l_p, given

$\in > 0$, $\exists$ a +ve integer m_0 such that n , $m \geq m_0$.

$$\Rightarrow \| x_n - x_m \|_p < \in \qquad (1)$$

In particular $$n \geq m_0 \Rightarrow \| x_n - x_{m_0} \|_p < \in \qquad (2)$$

Thus if $n \geq m_0$, then

$$\| x_n \|_p = \| x_n - x_{m_0} + x_{m_0} \|_p \leq \| x_n - x_{m_0} \|_p + \| x_{m_0} \|_p < \in + \| x_{m_0} \|_p$$

if $\in + \| x_{m_0} \|_p = A$ so that $A > 0$,

Then

$$\left[\sum_{n=1}^{\infty} | x_n^m |^p \right]^{\frac{1}{p}} < A$$

$$\| x_n \|_p < A \text{ for } A \geq m_0. \qquad (3)$$

As in the above examples, from (1), it can be shown that for fixed i, the sequence $< x_i^{(n)} >_{n=1}^{\infty}$ is a Cauchy sequence in C or R and consequently it must converge to a number say z_i.

Let $z = < z_1, z_2, \ldots\ldots., z_n >$ we assert that $z \in l_p$ and the cauchy sequence $< x_n >$ converges to $z \in l_p$ and we first show that $z \in l_p$, from (3) we have for $n \geq m_0$

$$\| x_n \|_p^p < A^p \Rightarrow \sum_{i=1}^{\infty} | x_i^{(n)} |^p < A^p$$

Hence for any +ve integer L, we have

$$\sum_{i=1}^{L} | x_i^{(n)} |^p < A^p \qquad (n \geq m_0) \qquad (4)$$

But for i = 1,….., L, we have $x_i^{(n)} \to z_i$ as $n \to \infty$. Hence letting $n \to \infty$ in (4), we obtain

$$\sum_{i=1}^{L} | z_i |^p \leq A^p (L = 1, 2, \ldots)$$

$$\Rightarrow \sum_{i=1}^{\infty} | z_i |^p \leq A^p < \infty$$

This proves that $z = < z_n >_{n=1}^{\infty}$ is in l_p.

Finally from (1), for n, $m \geq m_0$

$$\| x_n - x_m \|_p^p < \in^p \Rightarrow \sum_{i=1}^{\infty} | x_i^{(n)} - x_i^{(m)} |^p < \in^p$$

Hence for any +ve integer L, we have

$$\sum_{i=1}^{L} | x_i^{(n)} - x_i^{(m)} |^p < \in^p \qquad (n, m \geq m_0)]$$

Letting $m \to \infty$ and using $\lim_{m \to \infty} x_i^{(m)} = z_i$

we obtain

$$\sum_{i=1}^{L} | x_i^{(n)} - z_i |^p < \in^p \qquad \text{for all } n \geq m_0$$

Example 5: (The space l_2). Let l_2 denote the linear space of all sequences x = < $x_1, x_2, \ldots$> of all scalars such that

$$\sum_{n=1}^{\infty} | x_n |^2 < \infty$$

Show that l_2 is a Banach space under the norm $\| x \| = \left[\sum_{n=1}^{\infty} | x_n |^2 \right]^{\frac{1}{2}}$.

Solution: This space is called Hilbert coordinate space or sequence space.

This is a particular case of the previous example with p = 2. If the scalars are real, then l_2 is known as infinite dimensional Euclidean space and is denoted by R^{∞} . If the scalars are complex, then l_2 is called infinite dimensional unitary space denoted by C^{∞}.

Example 5. Let p be a positive real number. A measurable function f defined on [0, 1] is said to belong to the space $L^p = L^p [0, 1]$ if $\int_0^1 |f|^p < \infty$.

Thus L^1 consists precisely of the Lebesgue integrable functions on [0, 1]. Since $| f + g |^p \leq 2^p (| f |^p + | g |^p)$, it follows that $f + g \in L^p$ if $f, g \in L^p$. Also αf is in L^p, whenever f is and therefore $\alpha f + \beta g \in L^p$ whenever $f, g \in L^p$. For a function f in L^p, we define

$$\| f \| = \| f \|_p = \left(\int_0^1 | f |^p \right)^{\frac{1}{p}}$$

we observe that $\| f \| = 0 \Leftrightarrow f = 0$ almost everywhere. Thus one of the requirement for a space to be a normed linear space is not satisfied. To overcome this difficulty, we consider two measurable functions to be equivalent if they are equal almost every where. If we do not distinguish between equivalent functions, then L^p space shall become a normed linear space. Thus we should say that the elements of L^p are not functions but rather equivalence classes of functions.

If α is a constant, then $\| \alpha f \| = |\alpha| . \| f \|$. Thus to show that the linear space L^p is normed linear space, it is sufficient to show that $\| f + g \| \le \| f \| + \| g \|$. To show this again we establish two inequalities:

Holder's Inequality: If p and q are non – negative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $f \in L^p$ and $g \in L^q$, then $f g \in L^1$ and

$$\int |f g| \le \| f \|_p . \| g \|_q$$

Proof: The case p = 1 and q = 1 is straight forward. We assume therefore that $1 < p < \infty$ and consequently $1 < q < \infty$. Let us first suppose that

$\| f \|_p = \| g \|_q = 1.$ Using the inequality

$\alpha^{\lambda} \beta^{1-\lambda} \le \lambda \alpha + (1 - \lambda) \beta,$ α and β are non – negative reals.

Taking $\alpha = |f(t)|^p, \quad \beta = |g(t)|^q$

$\lambda = \frac{1}{p}, 1 - \lambda = 1 - \frac{1}{p} = \frac{1}{q},$ we obtain

$$|f(t) . g(t)| \le \frac{1}{p} |f(t)|^p + \frac{1}{q} |g(t)|^q$$

Now integration yields

$$\int |f g| \le \frac{1}{p} \int |f|^p + \frac{1}{q} \int |g|^q = 1 \qquad (1)$$

If $\| f \| = 0$ or $\| g \| = 0$, then the inequality to be established is trivial. Let f and g be any elements of L^p and L^q with $\| f \| \ne 0$. Then $\frac{f}{\| f \|_p}$ and $\frac{g}{\| g \|_q}$ both have norm 1. Substituting them in (1) gives

$$\frac{1}{\| f \|_p \| g \|_q} \int |fg| = \int \frac{|f|}{\| f \|_p} . \frac{|g|}{\| g \|_q} \le 1$$

and hence

$$\int |f g| \le \| f \|_p . \| g \|_q$$

Minkowski's Inequality: If f and g are in L^p, then so is f + g and

$$\| f + g \|_p \le \| f \|_p + \| g \|_p$$

Proof: Since $|f + g|^p \le 2^p (|f|^p + |g|^p)$, therefore $f, g \in L^p$ implies $f + g \in L^p$, the inequality is clear when p = 1, so we assume that p > 1. Let q > 1 such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(p - 1) q = p$. Also

$$\int |f+g|^p = \int |f+g|\,.\,|f+g|^{p-1}$$

$$\Rightarrow \int |f+g|^p \le \int |f|\,.\,|f+g|^{p-1} + \int |g|\,.\,|f+g|^{p-1} \qquad (1)$$

We note that

$$\int [|f+g|^{p-1}]^q = \int |f+g|^{(p-1)q} = \int |f+g|^p < \infty$$

since $p\,q - q = p$

Therefore $|f+g|^{p-1} \in L^q$. Since $f, g \in L^p$ and we have just shown that $|f+g|^{p-1} \in L^q$, Holder's inequality (proved above) implies that $|f|\,.\,|f+g|^{p-1}$ and $|g|\,.\,|f+g|^{p-1}$ are in L^1 and

$$\int |f|\,.\,|f+g|^{p-1} \le \|f\|_p\,.\,\|(|f+g|^{p-1})\|_q$$

$$\int |g|\,|f+g|^{p-1} \le \|g\|_p\,.\,\|(|f+g|^{p-1})\|_q$$

But, by definition of norm,

$$\|(|f+g|^{p-1})\|_q = \left\{\int |f+g|^{(p-1)q}\right\}^{1/q}$$

$$= \left\{\int |f+g|^p\right\}^{1/q}$$

$$= \left\{\|f+g\|_p^p\right\}^{1/q}$$

$$= \left\{\|f+g\|_p\right\}^{p/q}$$

Thus

$$\int |f|\,.\,|f+g|^{(p-1)} \le \|f\|_p \left\{\|f+g\|_p\right\}^{p/q} \qquad (2)$$

$$\int |g|\,.\,|f+g|^{(p-1)} \le \|g\|_p \left\{\|f+g\|_p\right\}^{p/q} \qquad (3)$$

Combining (1), (2) and (3), we have

$$\|f+g\|_p^p \le (\|f\|_p + \|g\|_p)\left\{\|f+g\|_p\right\}^{p/q}$$

Dividing throughout by $\{\|f+g\|_p\}^{p/q}$, we obtain

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

which completes the proof of Minkowski's inequality.
We have proved therefore that L^p space is a normed linear space. Now we prove that it is a complete space. We require some results.

A series $\sum f_n$ in a normed linear space is said to be summable to sum S if S is in the space and the sequence of partial sums of the series converges to S, that is,

$$\| S - \sum_{i=1}^{n} f_i \| \to 0$$

In such a case, we write $S = \sum_{i=1}^{\infty} f_i$. The series $\sum f_n$ is said to be absolutely summable if $\sum_{n=1}^{\infty} \| fn \| < \infty$.

We know that absolute convergence implies convergence in case of series of real numbers. This is not true in general for series of elements in a normed linear space. But this implication holds if the space is complete.

Completeness of L^p (Riesz – Fisher Theorem): For $1 \le p < \infty$, L^p – spaces are complete.

or

If $f_1, f_2, \ldots\ldots$ form a Cauchy sequence in L^p, that is $\| fn - f_m \|_p \to 0$ as $n, m \to \infty$ there is an $f \in L_p$ such that

$$\| fn - f \|_p \to 0.$$

Proof: To show that the Cauchy sequence $< fn >$ converges, we construct a subsequence of this sequence which converges almost every where on X as follows.

Since $< fn >$ is a Cauchy sequence, then for $\in = \frac{1}{2}$, $\exists$ a +ve integer n_1 s. that

$$n,\ m \ge n_1 \Rightarrow \| fn - f_m \|_p < \frac{1}{2}$$

Similarly for $\in = \left(\frac{1}{2}\right)^2$, we can choose a +ve integer $n_2 > n_1$ s. that $n, m \ge n_2$

$$\Rightarrow \| fn - f_m \|_p < \left(\frac{1}{2}\right)^2$$

In general having closed $n_1, \ldots, n_k$ let $n_{k+1} > n_k$ be s. that

$$\| fn - f_m \|_p < \left(\frac{1}{2}\right)^{k+1}$$

for all $n, m \ge n_{k+i}$ we assert that the subsequence $< f_{n_k} >_{k=1}^{\infty}$ converges a . e to a limit function, $f \in L_p$.

From the construction of $< f_{n_k} >$ it is evident that

$$\sum_{k=1}^{\infty} \| f_{n_{k+1}} - f_{n_k} \|_p < \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) = 1 \qquad (1)$$

If we define

$$g_k = | f_{n_1} | + | f_{n_2} - f_{n_1} | + \ldots\ldots + | f_{n_{k+1}} - f_{n_k} | = [f_{n_k}]$$

For k = 1, 2, 3,…… Then $< g_k >$ is an increasing sequence of non – negative measurable functions s. that

$$\| g_k^p \|_1 = \| g_k \|_p^p = [\| \{ | f_{n_1} | + | f_{n_2} - f_{n_1} | + \ldots\ldots + | f_{n_{k+1}} - f_{n_k} | \} \|_p]^p$$

$$\leq [\| f_{n_1} \|_p + \sum_{i=1}^{k} \| f_{n_{i+1}} - f_{n_i} \|_p]^p$$

by Minkowski's inequality.

$$\leq [\| f_{n_1} \|_p + \sum_{i=1}^{\infty} \| f_{n_{i+1}} - f_{n_i} \|_p]^p$$

$$< [\| f_{n_1} \|_p + 1]^p \qquad \text{by (1)}$$

$$< \infty \qquad \Rightarrow \| g_k^p \|_1 < \infty$$

or

$$\int | g_k |^p \, du < \infty$$

Let $g = \lim_{k\to\infty} g_k$. Then by Monotone convergence theorem and the above estimate of g_k^p, we have

$$\int | g |^p \, du = \lim_{k \to \infty} \int | g_k^p | \, du < \infty$$

i.e.

$$\int [| f_{n_1} | + \sum_{i=1}^{\infty} | f_{n_{i+1}} - f_{n_i} |]^p \, du < \infty \quad \text{Hence } g \in L_p.$$

It follows that

$$[| f_{n_1} | + \sum_{i=1}^{\infty} | f_{n_{i+1}} - f_{n_i} |]^p < \infty$$

a.e and so the series

$$\sum_{i=1}^{\infty} | f_{n_{i+1}}(x) - f_{n_i}(x) |$$

converges a. e and consequently the series

$$f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

Converges a.e. The k – th partial sum of this series is $f_{n_{k+1}}(x)$. and so the sequence $< f_{n_k}(x) >_{k=1}^{\infty}$
converges to a some non – negative measurable function f(x) for all $x \in A$ where A is measurable and u(A) = 0. Define f(x) = 0 for all $x \in A$. It is easy to see that f is measurable and complex valued on X.

We will now show that $f \in L_p$. Let $\in > 0$ be given. Choose l so large that

$$s, t \geq n_l \Rightarrow \| f_s - f_t \|_p < \in$$

Then for $k \geq l$ and $m > n_l$, we have

$$\| f_m - f_{n_k} \|_p < \in \Rightarrow \int | f_m - f_{n_k} |^p \, du^{\frac{1}{p}} < \in$$

$$\Rightarrow \int |f_m - f_{n_k}|^p \, du < \in^p \qquad (1)$$

By Fatou's Lemma, we have

$$\int |f - f_m|^p \, du = \int \lim_{k\to\infty} | f_{n_k} - f_m |^p \, du \leq \in^p \text{ by (2)}$$

Thus for each $m > n_l$, the function $f - f_m$ is in L_p and so $f = (f - f_m) + f_m$ is also in L^p and $\lim_{n\to\infty} \| f - f_n \|_p = 0$. Thus $f \in L_p$ is the limit of the sequence $< f_n >$. Hence L_p is complete.

Example 6: Consider the linear space of all n – tuples $x = (x_1, \ldots\ldots, x_n)$ of scalars and define the norm by

$$\| x \|_\infty = \max \{| x_1 |, | x_2 |, \ldots\ldots., | x_n |\} \ [\text{or } \sup | x_i |]$$

This space is denoted by l_∞^n.

Show that $(l_\infty^n, \| x \|_\infty)$ is a Banach space. (Also called the space of bounded sequence)

Solution: We first prove that l_∞^n is a normed linear space

[N_1] Since each $| x_n | \geq 0 \Rightarrow \| x \|_\infty \geq 0$

and $\| x \|_\infty = 0 \Leftrightarrow \max \{ | x_1 |, | x_2 |, \ldots., | x_n |\} = 0$

$$\Leftrightarrow | x_1 | = 0, | x_2 | = 0, \ldots\ldots\ldots., | x_n | = 0$$

$$\Leftrightarrow x_1 = 0, \ldots\ldots\ldots, x_n = 0$$

$$\Leftrightarrow (x_1, \ldots\ldots, x_n) = 0 \Leftrightarrow x = 0$$

[N_2] Let $x = (x_1, \ldots\ldots, x_n)$ and $y = (y_1, \ldots.., y_n)$

Then $\| x + y \|_\infty = \max \{ | x_1 + y_1 | , | x_2 + y_2 |, \ldots\ldots, | x_n + y_n |\}$.

$$\leq \max \{ |x_1| + |y_1|, |x_2| + |y_2|, \ldots, |x_n| + |y_n| \}$$

$$\leq \max \{ |x_1|, |x_2|, \ldots, |x_n| \} + \max \{ |y_1|, |y_2|, \ldots, |y_n| \}$$

$$= \|x\|_\infty + \|y\|_\infty.$$

[N_3] if α is any scalar, then

$$\|\alpha x\|_\infty = \max \{ |\alpha x_1|, |\alpha x_2|, \ldots, |\alpha x_n| \}$$

$$= \max \{ |\alpha|\,|x_1|, |\alpha|\,|x_2|, \ldots, |\alpha|\,|x_n| \}$$

$$= |\alpha| \max \{ |x_1|, |x_2|, \ldots, |x_n|$$

$$= |\alpha|\,\|x\|_\infty.$$

Hence l_∞^n is a normed linear space. We now show that it is a complete space. Let $< x_m >_{m-1}^{\infty}$ be any cauchy sequence in l_∞^n. Since each $x_m = < x_1^m, x_2^m, \ldots, x_n^m >$ Let $\in > 0$ be given, $\exists$ a +ve integer m_0 s.that $l, m \geq m_0$

$$\Rightarrow \|x_m - x_l\|_\infty < \in$$

$$\Rightarrow \max \{ |x_1^m - x_1^l|, |x_2^m - x_2^l|, \ldots, |x_n^m - x_n^l| < \in$$

$$\Rightarrow |x_i^{(m)} - x_i^{(l)}| < \in, \quad i = 1, \ldots, n.$$

This shows that for fixed i, $< x_i^{(m)} >_{m=1}^{\infty}$ is a Cauchy sequence of real (or complex) numbers. Since $\mathbb{C}$ or R is complete, it must converge to some $z_i \in \mathbb{C}$ or R. Thus the Cauchy sequence $< x_m >$ converges to $z = (z_1, z_2, \ldots, z_n)$. Rest of the proof is simple. Hence l_∞^n is a Banach space.

Show that l_∞ is a Banach space.

Example 7: Let C(X) denote the linear space of all bounded continuous scalar valued functions defined on a topological space X. Show that C(X) is a Banach space under the norm

$$\|f\| = \sup_{f \in C(X)} \{ |f(x)|, x \in X \}$$

Solution: Vector addition and scalar multiplication are defined by

$$(f + g)\, x = f(x) + g(x), \ (\alpha f)\, x = \alpha f(x)$$

C(X) is linear space under these operations. We now show that C(X) is a normed linear space.

$[N_1]$ Since $| f(x) | \geq 0 \ \forall \ x \in X$, we have

$$\| f \| \geq 0$$

and

$$\| f \| = 0 \Leftrightarrow \sup \{ | f(x) |, x \in X\} = 0$$

$$\Leftrightarrow | f (x) | = 0 \ \forall \ x \in X$$

$$\Leftrightarrow f(x) = 0 \ \forall \ x \in X$$

$$\Leftrightarrow f = 0 \qquad \text{(zero function).}$$

$[N_2]$

$$\| f + g \| = \sup \{ | (f + g)(x) | ; x \in X\}$$

$$= \sup \{ | f(x) + g(x) |; x \in X\}$$

$$\leq \sup \{ | f(x) | + | g(x) |; x \in X\}$$

$$\leq \sup \{ | f(x) |; x \in X\}$$

$$+ \sup \{ | g(x) | x \in X\}$$

$$= \| f \| + \| g \|$$

$[N_3]$

$$\| \alpha f \| = \sup \{ | (\alpha f) (x) |; x \in X\}$$

$$= \sup \{ | \alpha f(x) | ; x \in X\}$$

$$= \sup \{ | \alpha | | f(x) |; x \in X\}$$

$$= | \alpha |. \sup \{ | f(x) |; x \in X\}$$

$$= | \alpha | \| f \|.$$

Hence C(X) is a normed linear space. Finally we prove that C(X) is complete as a metric space. Let $< fn >$ be any Cauchy sequence in C(X). Then for a given $\in > 0$, $\exists$ a positive integer m_0 such that

$$m, n \geq m_0 \Rightarrow \| f_m - fn \| < \in$$

$$\Rightarrow \sup \{ | (f_m - fn)(x) |; x \in X \} < \in$$

$$\Rightarrow \sup \{ | f_m(x) - fn(x) |; x \in X \} < \in$$

$$\Rightarrow | f_m(x) - fn(x) | < \in \ \forall \ x \in X.$$

But this is the Cauchy's condition for uniform convergence of the sequence of bounded continuous scalar valued functions. Hence the sequence < fn > must converge to a bounded continuous function on X. It follows that C(X) is complete and hence it is a Banach space.

Continuous Linear Transformation

Definition: Let N and N′ be linear spaces with the same system of scalars. A mapping T from L into L′ is called a linear transformation if

$$T(x + y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

or equivalently $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

Also T(0) = T(0. 0) = 0 and

$$T(-x) = - T(x)$$

A linear transformation of one linear space into another is thus a homomorphism of first space into the second for it is a mapping which preserves the linear operations.

Definition: Let N and N′ be normed linear spaces with the same scalars and let T be a linear transformation of N into N′. We say that T is continuous, mean that it is continuous as a mapping of the metric space N into the metric space N′. [since every normed space is a metric space $d(x, y) = \| x - y \|$]. But by a result [Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous $\Leftrightarrow$ $x_n \to x \Rightarrow f(x_n) \to f(x)$.]

This implies that $x_n \to x$ in N $\Rightarrow T(x_n) \to T(x)$ in N′

In the next theorem, we convert the requirement of continuity into several more useful equivalent forms and show that the set of all continuous linear transformations of N into N′ can itself be made into a normed linear space in a natural way.

Theorem: Let N and N′ be normed linear spaces and T a linear transformation of N into N′. Then the following conditions on T are equivalent to one another.

(1) T is continuous

(2) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

(3) $\exists$ a real number $K \geq 0$ with the property that $\| T(x) \| \leq K \| x \|$ for every $x \in N$.

(4) If $S = \{ x : \| x \| \leq 1\}$ is the closed unit sphere in N, then the image T(S) is a bounded set in N′.

Proof: (i) $\Rightarrow$ (ii) If T is continuous, then by the property of linear transformation we have T(0) = 0 and it is certainly continuous at the origin. For if T is cont and $\{x_n\}$ is a sequence of points in N such that $x_n \to 0$, then by the continuity of T, we have

$$x_n \to 0 \Rightarrow T(x_n) \to T(0)$$

$$\Rightarrow T(x_n) \to 0 \quad \text{since } T(0) = 0.$$

Conversely if T is continuous at the origin and $\{x_n\}$ is a sequence such that $x_n \to x$, then

$$x_n \to x \Rightarrow x_n - x \to 0$$

$$\Rightarrow T(x_n - x) \to T(0) = 0 \qquad \text{[since T is continuous at the origin]}$$

$$\Rightarrow T(x_n) - T(x) \to 0$$

Hence T is continuous

(2) $\Rightarrow$ (3) Suppose that T is continuous at the origin. We shall show that $\exists$ a real number $K \geq 0$ such that $\| T(x) \| \leq K \| x \|$ for every $x \in N$.

We shall prove this result by contradiction. So suppose $\exists$ no such K. Therefore for each +ve integer n, we can find a vector x_n s. that

$$\| T(x_n) \| > n \| x_n \|$$

Which is equivalent to

$$\frac{\| T(x_n) \|}{n \| x_n \|} > 1 \quad \text{or} \qquad \| T\left(\frac{x_n}{n \| x_n \|}\right) \| > 1 \qquad (1)$$

we put $\qquad y_n = \dfrac{x_n}{n \| x_n \|}$

Then $\qquad \| y_n \| = \dfrac{\| x_n \|}{n \| x_n \|} = \dfrac{1}{n} \to 0$ as $n \to \infty$.

If follows from it that $y_n \to 0$. But from (1) $T(y_n) \not\to 0$. So T is not continuous at the origin which is contradiction to our assumption.

Conversely, suppose that $\exists$ a real number $K \geq 0$ with the property that $\| T(x) \| \leq K \| x \|$ for every $x \in N$. if $\{x_n\}$ is a sequence converging to zero, then

$$x_n \to 0 \Rightarrow \| x_n \| \to \| 0 \| = 0$$

Therefore $\qquad \| T(x_n) \| \leq K \| x_n \| \to 0$

And hence $T(x_n) \to 0$ which proves that T is continuous at the origin.

(3) $\Rightarrow$ (4) Suppose first that $\exists$ a real no $K \geq 0$ with the property that $\| T(x) \| \leq K \| x \|$ for every $x \in N$. If $S = \{x : \| x \| \leq 1\}$ is the closed unit sphere in N, then for all x, we have

$$\| T(x) \| \leq K \| x \|$$

$$\Rightarrow \| T(x) \| \leq K \;\forall\; x \in S.$$

Hence T(S) is a bounded set in N′.

Conversely suppose that $S = \{x : \| x \| \leq 1\}$ is the closed unit sphere in N and T(S) is bounded in N′. Then

$$\| T(x) \| \leq K \quad \forall\, x \in S$$

If x = 0, then T(x) = T(0) = 0 and therefore in this case we have clearly $\| T(x) \| \leq K \| x \|$. If $x \neq 0$, then $\frac{x}{\| x \|} \in S$ ($\because \| \frac{x}{\| x \|} \| = 1$) and therefore $\| T\left(\frac{x}{\| x \|}\right) \| \leq K$

$$\text{i.e. } \| T(x) \| \leq K \| x \| .$$

Space of Bounded Linear Transformation

Definition: A linear transformation T is said to be bounded if $\exists$ a non – negative real number K such that

$$\| T(x) \| \leq K \| x \| \; \forall\, x$$

K is called bound for T.

Remark: Thus according to the above theorem T is continuous iff it is bounded.

From condition (4) of our theorem, we can define the norm of a continuous linear transformation as follows:

Definition: Let T be a continuous linear transformation, then

$$\| T \| = \sup \{ \| T(x) \| ; \| x \| \leq 1\}$$

is called the norm of T.

Obviously norm of T is the smallest M for which $\| T(x) \| \leq M \| x \|$ holds for every

$$\text{i.e. } \| T \| = \text{Inf} \{M ; \| T(x)\| \leq M \| x \| \}$$

Theorem: Let N and N′ be normed linear spaces and let T be a linear transformation of N into N′. Then the inverse T^{-1} exists and is continuous on its domain of definition iff $\exists$ exists a constant m > 0 s. that

$$m \| x \| \le \| T(x) \| \; \forall \; x \in N. \qquad (1)$$

Proof: Let (1) hold. To show that T^{-1} exists and is continuous Now T^{-1} exists iff T is one – one. Let $x_1, x_2 \in N$. Then

$$T(x_1) = T(x_2) \Rightarrow T(x_1) - T(x_2) = 0 \qquad m \| x_1 - x_2 \| \le \| T(x_1 - x_2) \| = 0$$

$$\Rightarrow T(x_1 - x_2) = 0 \qquad \Rightarrow \| x_1 - x_2 \| = 0$$

$$\Rightarrow x_1 - x_2 = 0 \quad \text{by (1)}$$

$$\Rightarrow x_1 = x_2$$

Hence T is one one and so T^{-1} exists. Therefore to each y in the domain of T^{-1}, $\exists$ a x in N s. that

$$T(x) = y \Rightarrow x = T^{-1}(y) \qquad (2)$$

Hence (1) is equivalent to

$$m \| T^{-1} y \| \le \| y \| \Rightarrow \| T^{-1}(y) \| \le \frac{1}{m} \| y \|$$

$$\Rightarrow T^{-1} \text{ is bounded}$$

$$\Rightarrow T^{-1} \text{ is continuous (by the above theorem)}$$

conversely let T^{-1} exists and be continuous on its domain T[N]. Let $x \in N$. Since T^{-1} exists, there is an $y \in T[N]$ s. That

$$T^{-1}(y) = x \Leftrightarrow T(x) = y \qquad (3)$$

Again since T^{-1} is continuous, it is bounded so that $\exists$ a +ve constant K s. That

$$\| T^{-1} y \| \le K \| y \| \Rightarrow \| x \| \le K \| T(x) \| \text{ by (3)}$$

$$\Rightarrow m \| x \| \le \| T(x) \| \text{ where } m = \frac{1}{K} > 0$$

Theorem : Let N and N′ be normed linear spaces and let T be a bounded linear transformation of N into N′ : Put

$$a = \sup \{ \| T(x) \|; x \in N, \| x \| = 1\}$$

$$b = \sup \{ \| T(x) \| / \| x \|; x \in N; x \neq 0\}$$

$$c = \text{Inf} \{ K; K \geq 0, \| T(x) \| \leq K \| x \| \ \forall x \in N\}$$

Then

$$\| T \| = a = b = c$$

and

$$\| T(x) \| \leq \| T \| \, \| x \| \quad \forall x \in N.$$

Proof: By definition of norm

$$\| T \| = \sup \{ \| T(x) \|; x \in N, \| x \| \leq 1 \}$$

By definition of c, $\| T(x) \| \leq c \| x \| \ \forall x \in N$

and if $\| x \| \leq 1$, then $\| T(x) \| \leq c \ \forall x \in N$

and so $\sup \{ \| T(x) \|; x \in N, \| x \| \leq 1 \} \leq c$

i.e. $\| T \| \leq c$.

Also by definition of b and c, it is clear that $c \leq b$ [$\| T \| \leq c \leq b$]. Again if $x \neq 0$,

Then $\| T(x) \| / \|x \| = \left\| T \left(\frac{x}{\|x\|} \right) \right\|$

and $\frac{x}{\| x \|}$ has norm 1. Hence we conclude from the definitions of b and a that $b \leq a$. But it is evident that

$$a = \sup \{ \| T(x) \|; x \in N; \| x \| = 1 \} \leq \sup \{ \| T(x) \|; x \in N; \| x \| \leq 1 \}$$

$$\Rightarrow a \leq \| T \| .$$

Thus we have shown that

$$\| T \| \leq c \leq b \leq a \leq \| T \|$$

$$\Rightarrow \| T \| = a = b = c.$$

Finally definition of b shows that

$$\frac{\| T(x) \|}{\| x \|} \leq \sup \{ \frac{\| T(x) \|}{\| x \|}; x \in N, x \neq 0 \}$$

$$= b = \| T \|$$

$$\Rightarrow \| T(x) \| \leq \| T \| \| x \|$$

Remark : Now we shall denote the set of all continuous (or bounded) linear transformation of N into N′ by B(N, N′) [where letter B stands for bounded].

Theorem : If N and N′ are normed linear spaces, then the set B(N, N′) of all continuous linear transformation of N into N′ is itself a normed linear space with respect to the pointwise linear operations and the norm defined by

$$\| T \| = \sup \{ \| T(x) \|; \| x \| \leq 1 \}$$

Further if N′ is a Banach space, then B(N, N′) is also a Banach space.

Proof: Let B(N, N′) be the set of bounded linear transformation on N into N′. Let T_1, $T_2 \in$ B(N, N′). Define $T_1 + T_2$ by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)$$

and αT by

$$(\alpha T)(x) = \alpha T(x) \qquad \forall x \in N.$$

It can seen that under these operations of addition and scalar multiplication, B(N, N′) is a vector space since we know that the set S of all linear transformation from a linear space into another linear space is itself a linear space w.r.t. to the pointwise linear operations. Therefore in order to prove that B(N, N′) is a linear space, it is sufficient to show that B(N, N′) is a subspace of S. Let T_1, $T_2 \in B(N, N')$. Then T_1 and T_2 are bounded, so $\exists$ real numbers $K_1 \geq 0$ and $K_2 \geq 0$ s. that

$$\| T_1(x) \| \leq K_1 \| x \| \text{ and } \| T_2(x) \| \leq K_2 \| x \|$$

for all $x \in N$.

If α, β are any two scalars, then

$$\begin{aligned} \| (\alpha T_1 + \beta T_2)(x) \| &= \| (\alpha T_1)(x) + (\beta T_2)(x) \| \\ &= \| \alpha T_1(x) + \beta T_2(x) \| \\ &\leq | \alpha | \| T_1(x) \| + | \beta | \| T_2(x) \| \\ &\leq | \alpha | K_1 \| x \| + | \beta | K_2 \| x \| \\ &= [| \alpha | K_1 + | \beta | K_2] \| x \| \end{aligned}$$

Thus $\alpha T_1 + \beta T_2$ is bounded and so

$$\alpha T_1 + \beta T_2 \in B(N, N')$$

This proves that B(N, N′) is a linear subspace of S.

Now we prove that B(N, N′) is a normed linear space with respect to the norm defined by

$$\| T \| = \sup \{ \| T(x) \|; \| x \| \leq 1]$$

which is clearly non – negative. We have

(i) $$\| T \| = 0 \Leftrightarrow \sup \{ \| T(x) \|; \| x \| \leq 1 \} = 0$$

$$\Leftrightarrow \sup \{ \frac{\| T(x) \|}{\| x \|}, x \neq 0 \} = 0$$

$$\Leftrightarrow \frac{\| T(x) \|}{\| x \|} = 0 \ \forall \ x \in N, x \neq 0$$

$$\Leftrightarrow \| T(x) \| = 0$$

$$\Leftrightarrow T(x) = 0 \Leftrightarrow T = 0$$

(ii) $$\| \alpha T \| = \sup \{ \| (\alpha T)(x) \|; \| x \| \leq 1 \}$$

$$= \sup \{ \| \alpha . T(x) \|; \| x \| \leq 1 \}$$

$$= \sup \{ | \alpha | \| T(x) \|; \| x \| \leq 1 \}$$

$$= | \alpha | . \sup \{ \| T(x) \|; \| x \| \leq 1 \}$$

(iii) $$\| T_1 + T_2 \| = \sup \{ \| (T_1 + T_2) (x) \|; \| x \| \leq 1 \}$$

$$= \sup \{ \| T_1(x) + T_2(x) \}; \| x \| \leq 1 \}$$

$$\leq \sup \{ \| T_1(x) \|; \| x \| \leq 1 \}$$

$$+ \sup \{ \| T_2(x) \|; \| x \| \leq 1 \}$$

$$= \| T_1 \| + \| T_2 \|$$

Hence B(N, N′) is normed linear space. It remains to prove that if N′ is a Banach space, then B(N, N′) is also a Banach space. For if; suppose N′ is a Banach space. Then N′ is complete. It sufficies to show that B(N, N′) is complete. Let $\{T_n\}$ be an arbitrary cauchy sequence in B(N, N′), then for any $x \in N$,

$$\| T_m(x) - T_n(x) \| = \| (T_m - T_n) (x) \|$$

$$\leq \| T_m - T_n \| \| x \| \qquad (1)$$

$$[\because \| T(x) \| \leq \| T \| \| x\|]$$

This shows that $\{T_n(x)\}$ is a cauchy sequence in N′. Since N′ is complete, $\exists$ T(x) in N′ such that $T_n(x) \to T(x) \ \forall \ x \in N$ i.e.

$T(x) = \lim\limits_{n\to\infty} T_n(x)$. Now T defines a mapping T from N to N′.

It is obvious that T is linear. For

$$T(x + y) = \lim_{n\to\infty} T_n(x + y)$$

$$= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y)$$

$$= T(x) + T(y)$$

and $$T(\alpha. x) = \lim_{n\to\infty} T_n(\alpha x)$$

$$= \lim_{n\to\infty} \{ \alpha . T_n(x) \}$$

$$= \alpha . \lim_{n\to\infty} \{T_n(x) \} = \alpha . T(x)$$

Now $\{T_n\}$ being a cauchy sequence, $\lim_{m,n\to\infty} \{ \| T_n - T_m \| \} = 0$ and since

$$| (\| T_n \| - \| T_m \|) | \leq \| T_n - T_m \|$$

it follows that

$$\lim_{m,n\to\infty} | (\| T_n \| - \| T_m \|) | = 0$$

Therefore $[\| T_n \|]$ is convergent and hence bounded i.e. $\exists$ a real no K s. That

$$\| T_n \| \leq K, \qquad n = 1, 2, \ldots\ldots$$

and therefore $\quad \| T_n(x) \| \leq \| T_n \| \| x \| \leq K \| x \| \qquad \forall\ n$

Thus $\quad \| T(x) \| = \lim_{m,n\to\infty} \| T_n(x) \| \leq K \| x \|$

$$\Rightarrow T \text{ is bounded}$$

Hence $T \in B(N, N')$. If we prove that $T_n \to T$. Then we have that $B(N, N')$ is complete. For let $\in > 0$, choose n_0 so that

$$\| T_m - T_n \| < \frac{\in}{2} \text{ if } m, n > n_0. \text{ Then}$$

$$\| T_m(x) - T_n(x) \| < \frac{\in}{2} \| x \| \quad \text{for } m, n > n_0, x \in N.$$

Letting $n \to \infty$, we get

$$\| T_m(x) - T(x) \| < \frac{\in}{2} \| x \| \quad \text{for } m > n_0, x \in N$$

since $\quad T(x) = \lim_{n\to\infty} T_n(x).$

This implies that for $m > n_0$ and $\| x \| \leq 1$, we have

$$\| T(x) - T_n(x) \| = \| T(x) - T_m(x) + T_m(x) - T_n(x) \|$$

$$\leq \| T(x) - T_m(x) \| + \| T_m(x) - T_n(x) \|$$

$$\leq \| T(x) - T_m(x) \| + \| T_m - T_n \| \| x \|$$

$$\leq \| T(x) - T_m(x) \| + \| T_m - T_n \| \qquad [\because \| x \| \leq 1]$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

This shows that

$$\| T - T_n \| = \sup \{ \| T(x) - T_n(x) \|; \| x \| \leq 1 \} < \in$$

Hence $T_n \rightarrow T$.

Thus we have proved that B(N, N′) is a complete normed linear space.

Note: By the definition of bounded linear transformation, it is clear that a continuous linear transformation is bounded linear transformation and conversely.

Also if N and N′ are normed linear spaces, the space L(N, N′) or B(N, N′) is also called space of all continuous linear transformation. In notation if N = N′, the space is also denoted as B(N).

Definition: A continuous linear transformation of a normed linear space into itself is called operator on N. The normed linear space consisting of all linear operators on N is denoted by B(N) instead of B(N, N′). The above theorem asserts that if N is a Banach space. **B(N) is also a Banach Space.**

Definition: An algebra is a linear space whose vectors can be multiplied in such a way that

(i) $x(yz) = (xy)z$

(ii) $x(y + z) = xy + yz$ and $(x + y)z = xz + yz$

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all scalars α.

Thus an algebra is a linear space that is also a ring in which (iii) holds.

If the linear operators T_1 and T_2 are multiplied in accordance with the formula

$$(T_1 T_2)(x) = T_1(T_2(x)) \qquad \forall\, x \in N$$

Then ß (N) is a algebra in which multiplication is related to the norm by

$$\| T T' \| \leq \| T \| \, \| T' \|$$

This relation is proved by the following computation

$$\| T T' \| = \sup \{ \| (T T')(x) \|; \| x \| \leq 1 \}$$

$$= \sup [\| T(T'(x)) \|; \| x \| \leq 1 \}$$

$$\leq \sup \{ \| T \| \, \| T'(x) \|; \| x \| \leq 1 \}$$

$$= \| T \| \{ \sup \| T'(x) \|; \| x \| \leq 1 \}$$

$$= \| T \| \, \| T' \| \qquad (1)$$

Since we know that addition and scalar multiplication are joining continuous in normed linear space, they are also jointly continuous in ß (N). Also multiplication is continuous, since

$$\text{If } T_n \to T \text{ in } B(N) \text{ and } T_n' \to T' \text{ in } B(N)$$

Then $$T_n T_n' \to T T'$$

Since

$$\| T_n T_n' - T T' \| = \| T_n (T_n' - T') + (T_n - T) T' \|$$

$$\leq \| T_n \| \| T_n' - T' \| + \| T_n - T \| \| T' \|$$

But (T_n) being convergent sequence in ß (N), it must be bounded so $\exists$ M such that

$$\| T_n T_n' - T T' \| \leq M \| T_n' - T' \| + \| T' \| . \| T_n - T \| \to 0 \text{ as } n \to \infty.$$

We also remark that when $N \neq \{ 0 \}$ then the identity transformation I is an identity for the algebra ß(N). In this case we clearly have

$$\| I \| = 1$$

for $$\| I \| = \sup \{ \| I(x) \|, \| x \| = 1 \}$$

$$= \sup \{ \| x \|; \| x \| = 1 \}$$

$$= 1.$$

Definition: Let N and N′ be normed linear spaces. A one to one linear transformation T of N into N′ such that $\| T(x) \| = \| x \|$ for every x in N is called isometric isomorphism. N is said to be isometrically isomorphic to N′ if $\exists$ an isometric isomorphism of N onto N′.

Theorem: If M is a closed linear subspace of a normed linear space N and if T : $N \to N/M$ defined by $T(x) = x + M$. Show that T is continuous linear transformation for which $\| T \| \leq 1$.

Proof: Since M is closed, N/M is a normed linear space [since every closed subspace of normed space is normed] with the norm of a coset x + M in N/M defined by

$$\| x + M \| = \text{Inf} \{ \| x + m \|; m \in M \}$$

$$T(x_1 + x_2) = x_1 + x_2 + M$$

$$= x_1 + M + x_2 + M \qquad \text{[definition of N/M]}$$

$= T(x_1) + T(x_2)$

$T(\lambda x) = \lambda x + M = \lambda(x + M) = \lambda T.$

$\Rightarrow$ T is linear.

$\| T x \| = \| x + M \| = \text{Inf} \{ \| x + m \|; m \in M \}$

$\leq \text{Inf} \{ \| x \| + \| m \|; m \in M\}$

$\leq \text{Inf} \| x \| + \text{Inf} \| m \| ; m \in M$

$= \| x \| + 0.$

[since M is subspace of N, 0 is the element of M which has smallest norm namely zero]
Then

$\| T x \| \leq \| x \| \quad \forall n \in N$

$\Rightarrow$ T is bounded
Since

$$\sup_{x \neq 0} \frac{\| Tx \|}{\| x \|} \leq 1 \Rightarrow \| T x \| \leq \| x \| \leq 1 \Rightarrow \sup_{x \neq 0} \{ \| T(x) \|; \| x \| \leq 1\}$$

$\leq \| x \| \leq 1$

$\Rightarrow \| T \| \leq 1.$

Theorem: Let E and F be two normed linear spaces. Then they are topologically isomorphic iff $\exists$ m, M and a linear mapping $T : E \to F$ which is one-one and onto such that

$$m \| x \| \leq \| T x \| \leq M \| x \| \qquad \forall x \in E$$

Proof: Let E and F be top. isomorphic, then by definition $\exists$ linear mapping T : $E \to F$ such that T is cont, bijective and T^{-1} exists and is also continuous. Then by using theorem on continuous of linear transformation $\exists$ M such that

$$\| T x \| \leq M \| x \| \qquad \forall x \in E$$

Also by the last result, $\exists$ m > 0 such that

$$m \| x \| \le \| T x \| \le M \| x \|$$

Since T^{-1} exists and is continuous Then we have linear one – one onto mapping such that $\exists$ m > 0, M > 0 such that

$$m \| x \| \le \| T x \| \le M \| x \| \ \forall x \in E$$

conversely if $\exists$ T : E $\rightarrow$ F such that T is one – one onto and $\exists$ m , M s. That

$$m \| x \| \le \| T(x) \| \le M \| x \| \quad \forall x \in E.$$

Since $\| T(x) \| \le M \| x \|$

Hence T is bounded

By the theorem on continuity (or bounded) $\Rightarrow$ T is continuous.

Now From $m \| x \| \le \| T(x) \|$

T is 1 – 1 and onto $\Rightarrow$ T^{-1} exists.

$\Rightarrow$ T^{-1} is continuous. Hence T is bijective, cont and T^{-1} exists and is continuous or [T is open]

$\Rightarrow$ E and F are topologically isomorphic.

Remark: On a finite dimensional space $\mathfrak{R}^n$ or C^n, all the norms are equivalent in the sense that they define same topology up to top. isomorphism.

Definition: Let E and F be normed linear spaces. Then E and F are said to be equivalent as normed spaces iff $\exists$ m > 0, M > 0 such that

$$m \| x \| \le \| T x \| \le M \| x \| \quad \forall x \in E.$$

Conjugate of an Operator

Let N be a normed linear space and T a continuous linear operator on N. Then for any functional f $\in$ N*, the composite mapping (foT) is a continuous linear functional since

$$\begin{aligned} (foT)(\alpha x + \beta y) &= f(T(\alpha x + \beta y) \ ; x , y \in N \\ &= f(\alpha . T(x) + \beta (T(y)) \\ &= \alpha f(T(x)) + \beta f(T(y)) \\ &= \alpha (foT)(x) + \beta (f oT)(y) \end{aligned}$$

Moreover since f and T both are continuous, foT is also continuous Hence foT $\in$ N*.

Define a mapping

$$T^* : N^* \rightarrow N^*$$

by

$$T^*(f) = foT \qquad \forall\, f \in N^*.$$

This mapping is called the conjugate of the operator T.
Also we note that

$$(T^*(f))\,(x) = f(T(x)) \qquad \forall\, x \in N.$$

we assert that T* is linear, for

$$\begin{aligned}(T^*(\alpha\, f + \beta\, g)\,(x) &= (\alpha\, f + \beta\, g)\,(T(x))\\ &= \alpha\, f(T(x)) + \beta\, .\, g(T(x))\\ &= \alpha\,(f\,T)\,(x) + \beta\,(g\,T)\,(x)\\ &= \alpha(T^*(f))\,(x) + \beta\,(T^*(g))\,(x)\\ &= (\alpha\,(T^*(f) + \beta\,(T^*(g))\,(x)\end{aligned}$$

T* is also bounded (continuous) and hence

$$\begin{aligned}\|\, T^*\,\| &= \sup\,\{\,\|\, T^*\, f\,\|;\ \|\, f\,\| \le 1\,\}\\ &= \sup\,\{\,|\, T^*(f)\,(x)\,|;\ \|\, f\,\| \le 1 \text{ and } \|\, x\,\| \le 1\,\}\\ &= \sup\,\{\,|\, f(T(x))\,|;\ \|\, f\,\| \le 1,\ \|\, x\,\| \le 1\,\}\\ &\le \sup\,\{\,\|\, f\,\|\,\|\, T\,\|\,\|\, x\,\|;\ \|\, f\,\| \le 1,\ \|\, x\,\| \le 1\,\}\\ &\le \|\, T\,\| \qquad\qquad (1)\end{aligned}$$

Since N is a normed linear space, for a non – zero vector x in N, there exists a functional f on N such that

$$\|\, f\,\| = 1 \text{ and } f(T(x)) = \|\, T(x)\,\| \qquad [\because \|\, f\,\| = 1 \text{ and } f(x) = \|\, x\,\|]$$

Therefore

$$\begin{aligned}\|\, T\,\| &= \sup\,\{\,\|\, T\, x\,\|\,;\ \|\, x\,\| \le 1\,\}\\ &= \sup\,\{\, f(T(x))\,;\ \|\, x\,\| \le 1 \text{ and } \|\, f\,\| \le 1\,\}\end{aligned}$$

$$\leq \sup \{f(T(x)) \, ; \, \| x \| \leq 1 \text{ and } \| f \| \leq 1 \}$$

$$= \sup \{ \, | T^*(f)(x) | \, ; \, \| f \| \leq 1 \text{ and } \| x \| \leq 1 \}$$

$$= \sup \{ \, \| T^* f \| \, \| x \| \, ; \, \| f \| \leq 1 \text{ and } \| x \| \leq 1 \}$$

$$\leq \sup \{ \, \| T^* f \| \, ; \, \| f \| \leq 1 \}$$

$$= \| T^* \| \qquad (2)$$

From (1) and (2), it follows that

$$\| T \| = \| T^* \| \qquad (3)$$

consider the mapping

$$\phi : ß(N) \to ß(N^*)$$

defined by

$$\phi(T) = T^* \qquad \forall \; T \in ß(N)$$

Let $T_1, T_2 \in ß(N)$. Then

$$\phi(\alpha \, T_1 + \beta \, T_2) = (\alpha \, T_1 + \beta \, T_2)^*$$

But for all $f \in N^*$ and $x \in N$, we have

$$[(\alpha \, T_1 + \beta \, T_2)^* \, (f)] \, (x) = f \, [(\alpha \, T_1 + \beta \, T_2 \,) \, (x)]$$

$$= f \, [\, \alpha \, T_1(x) + \beta \, T_2(x) \,]$$

$$= \alpha \, f(T_1(x)) + \beta \, f \, (T_2(x))$$

$$= \alpha \, (f \, T_1) \, (x) + \beta \, (f \, T_2) \, (x)$$

$$= \alpha(T_1^*(f)) \, (x) + \beta \, (T_2^*(f))(x)$$

$$= (\alpha \, [T_1^*(f)] + \beta \, [T_2^*(f)]) \, (x)$$

$$= \{[\alpha \, T_1^* + \beta \, T_2^*] \, (f)\} \, (x)$$

Therefore, we have

$$\phi \, (\alpha \, T_1 + \beta \, T_2) = (\alpha \, T_1 + \beta \, T_2)^*$$

$$= \alpha \, T_1^* + \beta \, T_2^*$$

$$= \alpha \, \phi(T_1) + \beta \, \phi(T_2)$$

which shows that ϕ in linear.
Also ϕ is one to one, since

$$\phi(T_1) = \phi(T_2) \Rightarrow T_1^* = T_2^*$$

$$\Rightarrow T_1^*(f) = T_2^*(f) \qquad \forall\, f \in N^*$$

$$\Rightarrow [T_1^*(f)]\,(x) = [T_2^*(f)]\,(x)$$

$$\Rightarrow f(T_1(x)) = f(T_2(x))$$

$$\Rightarrow (T_1 - T_2)\,(x) = 0 \qquad \forall\, x \in N$$

$$\Rightarrow T_1 - T_2 = 0 \quad \Rightarrow T_1 = T_2$$

Moreover

$$\| \phi(T) \| = \| T^* \| = \| T \|$$

Hence ϕ is an isometric isomorphism and it preserves norm also.
If $f \in N^*$ and $x \in N$, then

$$\begin{aligned} [(T_1\, T_2)^*\,(f)]\,(x) &= f\,(T_1\, T_2)\,(x) \\ &= f\,(T_1\,(T_2(x)) \\ &= (f\, T_1)\,(T_2(x)) \\ &= (T_1^*(f))\,(T_2(x)) \\ &= T_2^*\,(T_1^*(f))\,(x) \\ &= [(T_2^*\, T_1^*)\,(f)]\,(x) \end{aligned}$$

i.e.

$$(T_1\, T_2)^* = T_2^*\, T_1^*$$

and if I is an identity operator, then

$$\begin{aligned} [\, I^*(f)]\,(x) &= f\,[I(x)\,] = f(x) \\ &= (I(f))\,(x) \\ &\Rightarrow I^* = I \end{aligned}$$

Thus we have proved the following:

Theorem: If T is an operator on a normed linear space N, Then its conjugate T* is defined by equation

$$[T^*(f)]\,(x) = f[T(x)]$$

is an operator on N* and the mapping $T \to T^*$ is an isometric isomorphism of ß(N) into ß(N*) which reverses the product and preserves the identity transformation.

Theorem: A non empty subset X of a normed linear space N is bounded $\Leftrightarrow$ f(X) is a bounded set of numbers for each f in N*.

Proof: Since $|f(x)| \le \|f\|\,\|x\|$, it follows that if X is bounded, then f(X) is also bounded for f.
To prove the converse, we write $X = \{x_i\}$. We now use natural imbedding $[x \to F_n]$ to map X to the subset $\{F_{x_i}\}$ of N**. The assumption that $f(X) = \{f(x_i)\}$ is bounded for each f implies that $\{F_{x_i}(f)\}$ is bounded for each f. Moreover since N* is complete. The uniform boundedness theorem shows that$\{F_{x_i}\}$ is a bounded subset of N**.
Since natural imbedding preserves norms, therefore X is evidently a bounded subset of N.

Conjugate Spaces

We know that the spaces R and C are real and complex complete normed linear spaces. If N is an arbitrary normed linear space, then the set **B**(N, R) or **B**(N, C) of all continuous linear transformations of N in R or C is a normed linear space. This space is called the conjugate space of N and is denoted by N*. The elements of N* are called continuous linear functionals or simply functionals. The norm of a function $f \in N^*$ is defined as

$$\|f\| = \sup \{|f(x)| \,;\, \|x\| \le 1\}$$

Since R and C are Banach spaces, it follows that B(N, R) and **B**(N, C) are also Banach spaces. Thus N* is also a Banach space.

Hahn-Banach Theorem and its applications

Hahn-Banach Theorem is a strong tool for functional analysis. In fact the theory of conjugate spaces rest on the Hahn-Banach Theorem which asserts that any linear functional on a linear subspace of a normed linear space can be extended linearly and continuously to the whole space without increasing its norm.

Statement of Hahn Banach Theorem : Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. Then f can be extended to a functional f_0 defined on the whole space N such that

$$f_0(x) = f(x)\ \forall\ x \in M \text{ and } ||f_0|| = ||f||$$

Proof :- Let f be a functional defined on a subspace M of a real normed linear space N and let x_0 be any vector of N which is not in M. Consider the set $\{M + tx_0\}$ of elements $x + tx_0$ where $x \in M$ and t is an arbitrary real number. Then $\{M + tx_0\}$ is obviously a linear manifold of N. Every element of $\{M + tx_0\}$ is uniquely representable in the form $x + tx_0$, for if 0 there exists two representations $y_1 = x_1 + t_1x_0$ and $y_2 = x_2 + t_2 x_0$, we can suppose that $t_1 \neq t_2$ for 0 otherwise $x_1 + t_1 x_0 = x_2 + t_2 x_0$ would imply $x_1 = x_2$ and the representation will be unique. Then

$$x_1 - x_2 = (t_2 - t_1)\, x_0$$

$$\Rightarrow \qquad x_0 = \frac{x_1 - x_2}{t_2 - t_1}$$

But this is impossible since $x_0 \notin M$ and $x_1, x_2 \in M$. Hence $t_1 = t_2$ and Thus $x_1 = x_2$ which proves the uniqueness.

For any two elements, $x_1, x_2 \in M$, we have

$$\begin{aligned} f(x_1) - f(x_2) &= f(x_1 - x_2) \\ &\leq |f(x_1 - x_2)| \\ &\leq ||f||.\ ||x_1 - x_2|| \\ &= ||f||.\ \{||x_1 + x_0 - (x_2 + x_0)||\} \\ &\leq ||f||\ \{||x_1 + x_0|| + ||x_2 + x_0||\} \end{aligned}$$

so that

$$f(x_1) - ||f||.\ ||x_1 + x_0|| \leq f(x_2) + ||f||.\ ||x_2 + x_0||$$

Since x_1 and x_2 are arbitrary in M,

We have

$$\sup_{x \in M}\{f(x) - ||f||.||x+x_0||\} \leq \inf_{x \in M}\{f(x) + ||f||.\ ||x+x_0||\}$$

Thus there exists a real no α which satisfies the inequality

$$\sup_{x \in M}\{f(x) - ||f||.\ ||x+x_0||\} \leq \alpha \leq \inf_{x \in M}\{f(x) + ||f||.||x+x_0||\} \qquad (1)$$

Now let y be an arbitrary element of $\{M + t\ x_0\}$. Then y is uniquely expressible in the form $y = x + tx_0$. We define a function ϕ on $\{M + tx_0\}$ by

$$\phi(y) = f(x) - t\alpha \ \forall\ y \in \{M + tx_0\}$$

where α is a fixed real number satisfying (1). Obviously ϕ coincides with f in M and the linearity of f implies that ϕ is linear. We shall show that ϕ in bounded and has the same norm as f(x). We distinguish two cases :

(i) $t > 0$. Since $\frac{x}{t} \in M$, the relation (1) yields

$$\begin{aligned} \phi(y) &= f(x) - t\alpha \\ &= t\left\{f\left(\frac{x}{t}\right) - \alpha\right\} \\ &\leq t\left\{\|f\| . \left\|\frac{x}{t} + x_0\right\|\right\} \\ &= \|f\| . \|x + tx_0\| \\ &= \|f\| . \|y\| \qquad \ldots(2) \end{aligned}$$

(ii) $t < 0$, In this case (i) yields

$$\begin{aligned} f\left(\frac{x}{t}\right) - \alpha &\geq -\|f\| . \left\|\frac{x}{t} + x_0\right\| \\ &= -\frac{1}{|t|} . \|f\| . \|y\| \\ &= \frac{1}{t} . \|f\| . \|y\| \end{aligned}$$

and therefore

$$\begin{aligned} \phi(y) &= f(x) - t\alpha \\ &= t\left\{f\left(\frac{x}{t}\right) - \alpha\right\} \\ &\leq t . \frac{1}{t} \|f\| . \|y\| \\ &= \|f\| . \|y\| \qquad \ldots(3) \end{aligned}$$

Thus from (2) and (3), it follows that

$$\phi(y) \leq ||f|| \,.\, ||y|| \ \forall \ y \in \{M + tx_0\}$$

Replacing y by –y in (2), we have

$$-\phi(y) \leq ||f|| \,.\, ||y|| \ \forall \ y \in \{M + tx_0\}$$

Therefore $|\phi(y)| \leq ||f|| \,.\, ||y|| \ \forall \ y \in \{M + tx_0\}$

and therefore $||\phi|| \leq ||f||$ …(4)

But ϕ being an extension of f from M to $\{M + tx_0\}$

we have $||\phi|| \geq ||f||$ …(5)

Hence from (4) and (5)

$$||\phi|| = ||f||$$

Now if the elements of the set N – M are arranged in transfinite sequence x_0, x_1, x_2,…, x_k,…, we extend the functional successively to the spaces

$\{M + tx_0\} = M_0$, $\{M_0 + tx_1\} = M_1$ and so on since the norm remains the same at each step, continuing the above process, we arrive at a functional f_0 which satisfies both the conditions, namely

$$f_0(x) = f(x) \ \forall \ x \in M \text{ and } ||f_0|| = ||f||$$

This completes the proof of the theorem.

Complex Form of Hahn Banach Theorem

When N is complex and f is a complex valued function defined on M, let f_1 and f_2 be the real and imaginary parts of f. Thus for each $x \in M$, we have

$$f(x) = f_1(x) + i\, f_2(x)$$

and

$$|f_1(x)|, |f_2(x)| \leq |f(x)| \leq ||f||. \, ||x||$$

we claim that f_1 and f_2 are real valued linear functionals. Let $\alpha \in R$ and consider

$$\alpha\, f(x) = \alpha f_1(x) + i\alpha\, f_2(x) \qquad \ldots(1)$$

Since f is a linear functional, (1) must equal

$$f_1(\alpha x) = \alpha f_1(x) \text{ and } f_2(\alpha x) = \alpha f_2(x)$$

In a similar fashion, we can show that sums are also preserved.

Now consider

$$i(f_1(x) + if_2(x)) = if(x) = f(ix) = f_1(ix) + if_2(ix)$$

Equating real and imaginary parts, we have

$$f_1(ix) = -f_2(x)$$

and $$f_2(ix) = f_1(x)$$

Thus $f(x) = f_1(x) - if_1(ix)$ …(2)

Now by the above proved theorem, there exists a function F_1 defined on the whole space extending f_1 such that

$$\|F_1\| = \|f_1\| \text{ and } F_1(x) = f_1(x) \ \forall \ x \in M$$

we now define

$$F(x) = F_1(x) - iF_1(ix) \qquad \text{…(3)}$$

We now assert that F extends f. To prove this let $x \in M$ and consider (3). Since F_1 extends f_1, so

$$F_1(x) = f_1(x) \text{ and } F_1(ix) = f_1(ix) = -f_2(x)$$

Thus

$$F(x) = f_1(x) + if_2(x) = f(x)$$

and hence F extends f.

Moreover by (3)

$$\begin{aligned} F(ix) &= F_1(ix) - iF_1(i^2x) \\ &= F_1(ix) - iF_1(-x) \\ &= F_1(ix) + i F_1(x) \end{aligned}$$

$$\begin{aligned} iF(x) &= i[F_1(x) - iF_1(ix)] \\ &= iF_1(x) + F_1(ix) \end{aligned}$$

we see that $F(ix) = iF(x)$

and therefore is a complex linear functional.

Put $F(x) = re^{i\theta}$, then

$$|F(x)| = |r\, e^{i\theta}| = r\, |e^{i\theta}|$$

$$= r = e^{-i\theta}.\ F(x)$$

Thus $F(e^{-i\theta}x)$ is a purely real quantity which implies that imaginary part of $F(e^{-i\theta}x)$ i.e.

$$-F_1(i\, e^{-i\theta}\, x) \text{ must be zero.}$$

Thus $\quad F(e^{-i\theta}x) = F_1(e^{-i\theta}\, x)$

and we have

$$|F(x)| = |F_1(e^{-i\theta}x)| \leq ||F_1||.\ ||x||.\ |e^{-i\theta}|$$

$$= ||f_1||.\ ||x||$$

$$= ||f||\ .\ ||x||$$

which gives $||F|| \leq ||f||$

Moreover F being an extension of f, we have

$$||F|| \geq ||f||$$

Hence $||F|| = ||f||$ and the proof is complete.

Applications of Hahn-Banach Theorem

Theorem 1:- In N is a normed linear space and x_0 is a non-zero vector in N, then there exists a functional f_0 in N* such that $f_0(x_0) = ||x_0||$ and

$||f_0|| = 1$. In particular if $x \neq y$ $(x, y \in N)$, there exists a vector $f \in N^*$ such that $f(x) \neq f(y)$.

Proof :- Consider the subspace

$$M = \{\alpha\, x_0\}$$

consisting of all scalar multiplies of x_0 and consider the functional f defined on M as follows :

$$f : M \rightarrow F,\ f(\alpha x_0) = \alpha.\ \|x_0\|$$

clearly, f is a linear functional with the property

$$f(x_0) = \|x_0\|$$

$$|f(\alpha x_0) = |\alpha|.\ \|x_0\|$$

$$= \|\alpha x_0\| \qquad \ldots(1)$$

$$\|f\| = \sup \{|f(\alpha x_0)| \ ;\ \|\alpha x_0\| \leq 1\}$$

$$= \sup \{\|\alpha x_0\| \ ;\ \|\alpha x\| \leq 1\}$$

$$\leq 1$$

But if there were a real constant k such that $k < 1$ and $|f(\alpha x_0)| \leq k\ \|\alpha x_0\|\ \forall\ \alpha x_0 \in M$. This will contradict the equality defined by (1). Thus $\|f\| = 1$. We have thus established that f is a bounded linear functional defined on the subspace M with norm 1. Now by Hahn-Banach Theorem, the functional f can extended to a functional f_0 in N* such that

$$f_0(x_0) = f(x_0) = \|x_0\| \text{ and } \|f_0\| = \|f\| = 1$$

This completes the proof.

In the particular case since $x \neq y$, $x - y \neq 0$ and so by the above, there exists an $f \in N^*$ such that

$$f(x-y) = \|x-y\| \neq 0$$

$$\Rightarrow \qquad f(x) - f(y) \neq 0$$

$$\Rightarrow \qquad f(x) \neq f(y).$$

Remark : (1) This result shows that N* separates the vectors of N.

(2) This result also shows that Hahn-Banach Theorem guarantee that any normed linear space has rich supply of functionals.

Theorem 2 :- Let M be a closed linear subspace of a normed linear space N and let ϕ be the natural mapping (homomorphism) of N onto N/M defined by $\phi(x) = x + M$. Show that ϕ is a continuous (or bounded) linear transformation for which

$$\|\phi\| \le 1.$$

Proof :- Since M is closed and N/M is a normed linear space with the norm of a coset x + M in N/M defined by

$$\|x + M\| = \text{Inf } \{\|x + m\| \, ; \, m \in M\}$$

ϕ is linear :- Let x, y be any two elements of N and α, β be any scalars. Then

$$\phi(\alpha x + \beta y) = (\alpha x + \beta y) + M = (\alpha x + M) + (\beta y + M)$$

$$= \alpha(x + M) + \beta\,(y + M)$$

$$= \alpha\,\phi(x) + \beta\phi(y)$$

$\Rightarrow \phi$ is linear.

ϕ is continuous :- $\|\phi(x)\| = \|x + M\|$

$$= \text{Inf } \{\|x + m\|; \, m \in M\}$$

$$\le \|x + m\| \quad \forall\, m \in M$$

In particular for m = 0, we have

$$\|\phi(x)\| \le \|x\| = 1. \ \|x\| \ \forall\, x \in N$$

It follows that ϕ is bounded by the bound 1 and consequently ϕ is continuous.

Further

$$\|\phi\| = \sup \{\|\phi(x)\| \, ; \, x \in N; \, \|x\| \le 1\}$$

$$\le \sup \{\|x\| \, ; \, x \in N; \, \|x\| \le 1\}$$

$$\le 1$$

Thus $\|\phi\| \le 1$.

Theorem 3:- Let M be a closed linear subspace a normed linear space N and let x_0 be a vector not in M, then there exists a functional F in N* such that

$$F(M) = \{0\} \text{ and } F(x_0) \ne 0$$

Proof :- Consider the natural map $\phi : N \to N/M$ defined by $\phi(x) = x + M$. As shown in the last theorem ϕ is a continuous linear transformation and if $m \in M$, then $\phi(m) = m + M = 0$, where 0 denotes the zero vector M in N/M.

In other words, $\phi(M) = \{0\}$

Also since $x_0 \notin M$, we have

$$\phi(x_0) = x_0 + M \neq 0.$$

Hence by theorem 1, there exists a functional $f \in (N/M)^*$ such that

$$f(x_0 + M) = \|x_0 + M\| \neq 0$$

We now define F by $F(x) = f(\phi(x))$.

Then F is a linear functional on N. With the desired properties as shown below :

F is linear :-

$$F(\alpha x + \beta y) = f(\phi(\alpha x + \beta y)) = f(\alpha x + \beta y + M)$$

$$= f(\alpha(x + M) + \beta(y + M))$$

$$= \alpha f(x + M) + \beta f(y + M)$$

$$= \alpha f(\phi(x)) + \beta f(\phi(y))$$

$$= \alpha . F(x) + \beta . F(y)$$

F is bounded :-

$$|F(x)| = |f(\phi(x)|$$

$$\leq \|f\| . \|\phi(x)\|$$

$$\leq \|f\| . \|\phi\| . \|x\|$$

$$\leq \|f\| . \|x\|$$

since $\| \phi \| \leq 1$

Since f is bounded (being a functional). It follows from the above inequality that F is bounded. Thus F is a functional on N i.e. $F \in N^*$. Further if $m \in M$, then

$$F(m) = f(\phi(m)) = f(0) = 0$$

Thus $F(M) = 0 \ \forall \ m \in M$

and $F(x_0) = f(\phi(x_0)) = f(x_0 + M) \neq 0$

Theorem 4 :- Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M. If d is the distance from x_0 to M, show that there exists a functional $f_0 \in N^*$ such that

$$f_0(M) = \{0\},\ f_0(x_0) = d \text{ and } ||f_0|| = 1.$$

Proof :- Since by definition

$$d = \text{Inf}\ \{||x_0 + m||\ ;\ m \in M\}$$

Since M is closed and $x_0 \notin M \Rightarrow d > 0$.

Now consider the subspace

$$M_0 = \{x + \alpha x_0\ ;\ x \in M \text{ and } \alpha \text{ real}\}$$

Spanned by M and x_0. Since $x_0 \notin M$, the representation of each vector y in M_0 in the form $y = x + \alpha\, x_0$ is unique. For if there exists two scalars α_1 and α_2 and vectors x_1 and x_2 in M such that

$$y = \alpha_1\, x_0 + x_1 \text{ and } y = \alpha_2\, x_0 + x_2$$

$$\Rightarrow \qquad (\alpha_1 - \alpha_2)\, x_0 = x_2 - x_1$$

$$\Rightarrow \qquad x_0 = \frac{x_2 - x_1}{\alpha_1 - \alpha_2}$$

$\Rightarrow$ $x_0 \in M$ which is a contradiction, since $x_0 \notin M$ by our assumption. So each y in M_0 is unique. Define the map $f : M_0 \rightarrow R$ by

$$f(y) = \alpha d$$

where $y = x + \alpha x_0$ and d as in hypothesis. Because of the uniqueness of y, the mapping f is well defined. Also f is linear on M_0, and

$$f(x_0) = f(0 + 1.\ x_0) = 1.\ d = d \text{ and if } m \in M,$$

then $\qquad f(m) = f(m + 0.\ x_0) = 0.\ d = 0$

so that $\qquad f(M) = \{0\}$.

We now prove that $||f|| = 1$.

Since

$$||f|| = \sup_{||y||=1} \left\{ \frac{|f(y)|}{||y||} ; y \in M_0 y \neq 0 \right\}$$

$$= \sup \left\{\frac{|f(x+\alpha x_0)|}{\|x+\alpha x_0\|}; x \in M, \alpha \in R\right\}$$

$$= \sup \left\{\frac{|\alpha d|}{\|x+\alpha x_0\|}; x \in M; \alpha \in R, \alpha \neq 0\right\}$$

$$= \sup \left\{\frac{d}{\left\|x_0 + \frac{x}{\alpha}\right\|}; x \in M, \alpha \in R, \alpha \neq 0\right\}$$

$$= d \sup \left\{\frac{1}{\|x_0 - z\|}; z = -\frac{x}{\alpha} \in M\right\}$$

$$= d[\text{Inf}\,\{\|x_0 - z\|;\, z \in M\}]^{-1}$$

$$= d \cdot \frac{1}{d}$$

$$= 1.$$

Thus f is a linear functional on M_0 such that

$$f(M) = \{0\},\ f(x_0) = d \text{ and } \|f\| = 1. \qquad \ldots(*)$$

Hence by Hahn Banach Theorem, there exists a functional f_0 on the whole space N such that

$$f(y) = f_0(y)\ \forall\ y \in M_0 \text{ and } \|f\| = \|f_0\|$$

Thus from $(*)$

$$f_0(M) = \{0\},\ f_0(x_0) = d \text{ and } \|f_0\| = 1.$$

Riesz – Representation Theorem for Bounded Linear Functionals on L^p

Let F be a bounded linear function on L^p, $1 \le p < \infty$. Then there is a function g in L^q such that

$$F(f) = \int f\,g,\quad f \in L_p \text{ is arbitrary.}$$

Proof: Let F be a bounded linear functional on L^p, $1 \le p < \infty$. We put

$$\chi_S(x) = \begin{cases} 1 \text{ for } 0 \le x < S \\ 0 \text{ for } S \le x \le 1 \end{cases}$$

and show that

$$\Phi(S) = F(\chi_S(x))$$

is absolutely continuous For this purpose, let $\{(S_i, t_i)\}$ be any finite collection of non – overlapping subintervals of [0, 1] of total length less than δ.
Then

$$\sum_{i=1}^{n} | \Phi(t_i) - \Phi(S_i) |$$

$$= \sum_{i=1}^{n} \frac{|\Phi(t_i) - \Phi(S_i)|}{[\Phi(t_i) - \Phi(S_i)]} [\Phi(t_i) - \Phi(S_i)]$$

$$= \sum_{i=1}^{n} \text{sgn } [\Phi(t_i) - \Phi(S_i)] [\Phi(t_i) - \Phi(S_i)]$$

$$= F\{ \sum_{i=1}^{n} \text{sgn } [\chi_{t_i}(x) - \chi_{S_i}(x)] [\chi_{t_i}(x) - \chi_{S_i}(x)]\}$$

$$\le \| F \| \, \| \sum_{i=1}^{n} \text{sgn } [\chi_{t_i}(x) - \chi_{S_i}(x)] [\chi_{t_i}(x) - \chi_{S_i}(x)]\} \|$$

$$= \| F \| \, \{ \int_0^1 | \sum_{i=1}^{n} \text{sgn } [\chi_{t_i}(x) - \chi_{S_i}(x)] [\chi_{t_i}(x) - \chi_{S_i}(x)] |^p \, dx \}^{1/p}.$$

If we take $\delta = \frac{\in^p}{\| F \|^p}$, then it follows that total variation of Φ is less than $\in$ over any finite collection of disjoint intervals of total length less than δ. Thus Φ is absolutely continuous
Also we know that a function F is absolutely continuous iff it is indefinite integral. Therefore $\exists$ an integrable function g such that

$$\Phi(S) = \int_0^S g$$

Thus

$$F(\chi_S) = \int_0^1 g \, \chi_S \qquad \text{where } \chi_S = \begin{cases} 1 \; if \; x \in S \\ 0 \; if \; x \notin S \end{cases}$$

Since every step function on [0, 1] is [equal except at a finite number of pts to] to a suitable linear combination $\Sigma c_i \, \chi_{S_i}$, we must have

$$F(\psi) = \int_0^1 g \, \psi \qquad (*)$$

For each step function ψ by the linearity of F and of the integral.
Let f be any bounded measurable function on [0, 1] [hence Lebesgue integrable]. Then it follows that there is a sequence $< \psi_n >$ of step functions which converges almost everywhere to f. Since the sequence $< | f - \psi_n |^p >$ is

uniformly bounded and tends to zero almost every where by the bounded convergence theorem [Let < fn > be a sequence of measurable functions defined on a set E os finite measure and suppose that there is a real number M s. that

$| fn(x) | \leq M$ for all n and all x . If f(x) = lim fn (x) for each x in E, then $\int_E f = \lim \int_E fn$] implies that $\| f - \psi_n \|_p \to 0$ Since F is bounded and

$$| F(f) - F(\psi_n) | = | F(f - \psi_n) | \leq \| F \| \| f - \psi_n \|_p$$

we must have

$$F(f) = \lim F(\psi_n) \qquad (**)$$

Since g ψ_n is always less than | g | times the uniform bound for the sequence < ψ_n >, we have

$$\int f g = \lim \int g \psi_n \qquad (***)$$

by the Lebesgue convergence theorem (Let g be integrable over E and let < fn > be a sequence of measurable functions such that $| fn | \leq g$ on E and for almost all x in E we have f(x) = lim fn(x)

Then $$\int_E f = \lim \int_E fn.$$

Consequently, we must have

(*) $$\int f g = F(f) \quad \text{using } (***) , (*), (**)$$

for each **bounded measurable function f.** Since

$$| F(f) | \leq \| F \| \| f \|_p,$$

we have g in L_q and $\| g \|_q \leq \| F \|$ by the Lemma which states that "Let g be an integrable function on [0, 1] and suppose that there is a constant M such that $| \int f g | \leq M \| f \|_p$ for all bounded measurable function f. then g is in L^q and $\| g \|_q \leq M$" thus we have only to show that F(f) = $\int f g$ for each f in L^p. Let f be an arbitrary function in L^p. Then there is for each $\in > 0$, a step function ψ such that $\| f - \psi \|_p < \in$. Since ψ is bounded , we have

$$F(\psi) = \int \psi g$$

Hence

$$| F(f) - \int f g | = | F(f) – F(\psi) + \int \psi g - \int f g |$$

$$\leq | F(f - \psi) | + | \int (\psi - f) g |$$

$$\leq \| F \| \| f - \psi \|_p + \| g \|_q \| f - \psi \|_p$$

$$< [\| F \| + \| g \|_q]\in.$$

Since $\in$ is an arbitrary number, we must have

$$F(f) = \int f g$$

Riesz – Representation theorem for bounded linear functional on C[a, b].

Theorem: Let $F \in C^*[a, b]$. Then there exists a function $g \in B V [a, b]$ [bounded variation] such that for all $f \in C[a, b]$.

$$F(f) = \int_a^b f(t)\, dg(t)$$

Such that

$$\| F \| = V(g)$$

where V(g) denotes the total variation of g(t).

Proof: If we view C[a, b] as a subspace of B[a, b], by Hahn – Banach theorem, there exists a bounded linear functional F_0 defined on all of B[a, b], defined extending F and such that $\| F_0 \| = \| F \|$. Define the characteristic function

$$\chi_t (x) = \begin{cases} 1 \text{ for } a \leq x \leq t \\ 0 \text{ for } t \leq x \leq b \end{cases}$$

Obviously, for each such t,

$$\chi_t(x) \in B[a, b]$$

with F_0 the extension of F, we now define a function g(t) by

$$F_0[\chi_t(x)] = g(t).$$

We partition the interval [a, b] into

$$a = t_0 < t_1 < \ldots\ldots\ldots < t_n = b$$

and consider the sum

$$\sum_{i=1}^{n} | g(t_i) – g(t_{i-1}) |.$$

Putting

$$\in_i = \text{sgn}\,[g(t_i) - g(t_{i-1})] = \frac{|\,g(t_i) - g(t_{i-1})\,|}{[g(t_i) - g(t_{i-1})]}$$

we obtain

$$\sum_{i=1}^{n} |\,g(t_i) - g(t_{i-1})\,| = \sum_{i=1}^{n} \in_i [\,g(t_i) - g(t_{i-1})\,]$$

$$= \sum_{i=1}^{n} \in_i [\,F_0(\chi_{t_i}) - F_0(\chi_{t_{i-1}}\ 0)$$

$$= F_0 [\sum_{i=1}^{n} \in_i (\chi_{t_i} - \chi_{t_{i-1}})]$$

Therefore

$$\sum_{i=1}^{n} |\,g(t_i) - g(t_{i-1})\,| \le \|\,F_0\,\|\ \|\sum_{i=1}^{n} \in_i (\chi_{t_i} - \chi_{t_{i-1}})\,\|$$

$$= \|\,F\,\|$$

because $\|\,F_0\,\| = \|\,F\,\|$ and $\|\sum_{i=1}^{n} \in_i (\chi_{t_i} - \chi_{t_{i-1}})\,\| = 1$

Hence

$$|\sum_{i=1}^{n} |\,g(t_i) - g(t_{i-1})\,| \le \|\,F\,\|$$

that is g(t) is of bounded variation.
Also it follows that

$$V(g) \le \|\,f\,\| \qquad (1)$$

Suppose now that $f \in C[a, b]$ and define

$$Z_n(t) = \sum_{i=1}^{n} f(t_i)\,[\,\chi_{t_i}(x) - \chi_{t_{i-1}}(x)]$$

Where the sequence $< Z_n - (t) >$ converges strongly to f(t) i.e. $\|\,Z_n - f\,\| \to 0$.
Then the equality,

$$F_0(Z_n) = \sum_{i=1}^{n} f(t_i)\,[\,g(t_i) - g(t_{i-1})\,]$$

Implies that

$$\lim_{n \to \infty} F(Z_n) = \lim_{n} \sum_{i=1}^{n} f(t_i)\,[\,g(t_i) - g(t_{i-1})\,]$$

$$= \int_a^b f(t)\,dg(t)$$

by the definition of Riemann – Stieltjes integral. Since the sequence < $Z_n(t)$ > converges strongly to f(t) i.e. $\| Z_n - f \| \to 0$ and F_0 is a **bounded** (or continuous) linear functional and therefore cont, this implies that

$$F_0(Z_n) \to F_0(f)$$

Therefore

$$F_0(f) = \int_a^b f(t)\, dg(t).$$

Now since f was an arbitrary continuous function on [a, b] and F_0 must agree with F on C[a, b], we can write

$$F(f) = \int_a^b f(t)\, dg(t) \quad \text{for any } f \in C[a, b] \qquad (2)$$

From (2), we have

$$| F(f) | = \left| \int_a^b f(t)\, dg(t) \right|$$

$$\leq \max_{t \in [a,b]} | f(t) | \,.\, V(g).$$

$$= \| f \|\, V(g)$$

$$= \| f \|\, V(g) \qquad \text{for all } f \in C[a, b]$$

Taking sup $\| f \| \leq 1$, we have

$$\| F \| \leq V(g) \qquad (3)$$

From (1) and (3), it follows that

$$\| F \| = V(g).$$

Unit-III

Second Conjugate Spaces

We know that the conjugate space N* of a normed linear space N is itself a normed linear space. As R and C are normed linear spaces, we can form the conjugate space (N*)* of N* and denote this by N** and call it the second conjugate or dual space of N. The importance of N** lies in the fact that each vector x in N give rise to a functional F_x in N** and that there exists an isometric isomorphism of N into N** called the natural imbedding of N into N**.

The following definition will be required to establish natural imbedding of N in N**.

Definition :- Let N and N′ be normed linear spaces. Then a one to one linear transformation T : N→N′ of N in N′ is called isometric isomorphism of N into N′ if

$$\|Tx\| = \|x\| \text{ for every x in N.}$$

Further if there exists an, isometric isomorphism of N onto N′, then N is said to isometrically isomorphic to N′.

We now show that to each vector x∈N, there is a functional F_x in N**. Hence we prove the following result.

Theorem :- Let N be an arbitrary normed linear space. Then for each vector x∈N, the scalar valued function F_x defined by

$$F_x(f) = f(x) \qquad \forall\ f \in N^*$$

is a continuous linear functional in N** and the mapping $x \to F_x$ is then an isometric isomorphism of N into N**.

Proof :- Let N be an arbitrary normed linear space. Let x be a vector in N, consider the scalar valued function F_x defined by

$$F_x(f) = f(x) \qquad \forall\ f \in N^*$$

We assert that **F_x is linear.** In fact

$$F_x(\alpha f + \beta g) = (\alpha f + \beta g)\,(x)$$

$$= \alpha\ f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$$

Now computing the norm of F_x, we have

$$\|F_x\| = \sup\ \{|F_x(f)|\ ;\ \|f\| \le 1\}$$

$$= \sup\ \{|f(x)|\ ;\ \|f\| \le 1\}$$

$$\le \sup\ \{\|f\|\ \|x\|\ ;\ \|f\| \le 1\}$$

$$\le \|x\| \qquad \ldots(1)$$

Therefore **F_x is bounded** and a continuous linear functional on N*. [F_x is called the functional on N* induced by the vector x and is referred to as induced functional]

Now define a mapping $\phi : N \to N^{**}$

by $\qquad \phi(x) = F_x \qquad \forall\ x \in N.$

Clearly ϕ is one to one, since

$$\phi(x) = \phi(y) \Rightarrow F_x = F_y$$

$\Rightarrow \qquad F_x(f) = F_y(f) \quad \forall\ f \in N^*$

$\Rightarrow \qquad f(x) = f(y)$

$\Rightarrow \qquad f(x-y) = 0 \ \ \Rightarrow x-y = 0 \ \ \Rightarrow x = y.$

Let x, $y \in N$, then for all scalars α and β,

$$\phi(\alpha x + \beta y) = F_{\alpha x+\beta y}$$

If $f \in N^*$ then

$$F_{\alpha x+\beta y}(f) = f(\alpha x+\beta y)$$

$$= \alpha f(x) + \beta f(y)$$

$$= \alpha\ F_x(f) + \beta(F_y(f)$$

$$= (\alpha F_x)\ (f) + (\beta F_y)\ (f)$$

$$= (\alpha\ F_x + \beta\ F_y)\ (f)$$

$$= \alpha\ F_x + BF_y$$

Thus

$$F_{\alpha x+\beta y} = \alpha F_x + \beta F_y$$

and hence

$\Rightarrow$ $$\phi(\alpha x + \beta y) = \alpha F_x + \beta F_y = \alpha \phi(x) + \beta\phi(y)$$

which shows that ϕ is linear

Moreover by (1)

$$\|\phi(x)\| = \|F_x\| \leq \|x\| \qquad \ldots(2)$$

Also we know that if x is a non-zero vector in N, then there exists a functional f_0 in N* such that $f_0(x) = \|x\|$ and $\|f_0\| = 1$. So

$$\| x \| = f_0(x) \leq \sup \{|f_0(x)| ; f_0 \in N^* \text{ and } \|f_0\| = 1\}$$

$$= \sup\{|F_x(f_0)| ; \|f_0\| = 1 \}$$

$\Rightarrow$ $$= \| \phi(x) \| \qquad \ldots(3)$$

$$\| x \| \leq \| \phi(x) \|$$

Thus from (2) and (3)

$$\|\phi(x)\| = \|x\| \quad \forall\, x \in N.$$

$\Rightarrow$ ϕ is an isometry.

It follows therefore that $x \to F_x$ is an isometric isomorphism of N into N**.

Remark :- This isometric isomorphism is called the **natural imbedding** of N into N**, for we may regard N as a part' of N** without altering any of its structure as a normed linear space and we write

$$N \subseteq N^{**} .$$

Reflexive Spaces

Definition :- A normed linear space N is said to be reflexive if N = N**

The space l_p and l_q for $1 < p < \infty$ are reflexive since $l_p^* = l_q$ $\Rightarrow$

$$l_p^{**} = l_q^* = l_p.$$

Remark :- Every reflexive space is a Banach space since N** is a complete space. But a Banach space may be non-reflexive space for ex. C[0, 1] is a Banach space but it is not reflexive.

Example :- $(l_p^n)^* = l_q^n$

$$(l_1^n)^* = l_\infty^n, \qquad (l_\infty^n)^* = l_1^n$$

where

$$l_p^n = \left\{ x = (x_1, x_2, ..., x_n), \| x \| = \left(\sum_1^n | x_i |^p \right)^{1/p} \right\}$$

$$l_1^n = \left\{ x = (x_i)_{i=1}^n ; \| x \| = \sum_1^n | x_i | \right\}$$

$$l_\infty^n = \{ x = (x_i)_{i=1}^n ; \| x \| = \max_{1 \le i \le n} |x_i| \}$$

Solution :- Let L be the linear space of n–triples $x = (x_1, x_2, \ldots, x_n)$.

If $\{e_1, e_2, \ldots, e_n\}$ is a natural basis of L

$$[e_1 = (1, 0, 0, \ldots.)\ e_2 = (0, 1, \ldots)\ e_3 = (0, 0, 1, \ldots)]$$

Then $x = x_1e_1 + x_2e_2 + \ldots + x_ne_n$

If f is any linear functional on L i.e. A scalar valued linear function

$$f(x) = f(x_1e_1 + \ldots + x_ne_n)$$

$$= f(x_1e_1) + \ldots + f(x_n e_n)$$

$$f(x) = x_1 f(e_1) + \ldots + x_n f(e_n)$$

where x_i′s are scalars.

Put $f(e_1) = y_1 \ldots, f(e_n) = y_n$, then

$(y_1, \ldots, y_n)$ is an n-tuples of scalars. Thus

$$f(x) = \sum_{i=1}^n x_i y_i \ \forall\ x = (x_i)_1^n \in L.$$

is a linear functional

since $f(x + x') = \sum_{1}^{n} (x_i + x_i') y_i$

$$= \sum_{1}^{n} (x_i y_i + x_i' y_i)$$

$$= \sum_{1}^{n} x_i y_i + \sum_{1}^{n} x_i' y_i$$

$$= f(x) + f(x')$$

Similarly $f(\alpha x) = \sum_{1}^{n} \alpha x_i y_i = \alpha \sum_{1}^{n} x_i y_i = \alpha f(x) \quad \forall \alpha$ scalar.

Thus we have a 1–1, onto mapping defined by

$$y = (y_1, y_2, \ldots, y_n) \to F$$

where $f \in L^*, y \in L$

Thus algebraically $L' = L$

By defining a suitable norm, say the norm

$\|x\| = \left(\sum_{1}^{n} | x_i^p |\right)^{1/p}$ on L to make it l_p^n space, the L' space of all continuous functionals is equal to $(l_p^n)^*$, where the norm of f is given by

$$\|f\| = \text{Inf } \{k \; ; \; k \geq 0 \text{ and } |f(x)| \leq k \|x\|\} \quad \Rightarrow x \in l_p^n.$$

It is sufficient to show that what norm of $y = (y_1, \ldots, y_n)$ makes the mapping $y \Leftrightarrow f$ an isometric isomorphism].

Case I :- when $1 < p < \infty$,

Then we can show that $(l_p^n)^* = l_q^n$

$$\|x\| = \left(\sum_{1}^{n} | x_i |\right)^{1/p} \quad \forall x \in l_p^n.$$

If f is continuous linear functional

$$|f(x)| = \left| \sum_{1}^{n} x_i y_i \right|$$

$$\leq \sum_{1}^{n} |x_i\, y_i|$$

$$\leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

[By using Holder's inequality]

$$|f(x)| \leq \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \|x\|$$

Thus we have

$$\|f\| \leq \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

since $|f(x)| \leq \|f\|\, \|x\|$

$$\left[\|f\| = \text{Inf}\left[\sum_{i=1}^{n} |y_i|^q\right]^{1/q}\right]$$

For the other inequality consider the vector x defined by

$$x_i = 0 \text{ if } y_i = 0$$

and $\quad x_i = \dfrac{|y_i|^q}{y_i}$ otherwise.

$$f(x) = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} |y_i|^q$$

$$\frac{|f(x)|}{\|x\|} = \frac{\sum_{i=1}^{n} |y_i|^q}{\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}}$$

$$= \frac{\sum_{i=1}^{n} |y_i|^q}{\left(\sum_{i=1}^{n} |y_i|^{p(q-1)}\right)^{1/p}} \qquad \text{since } |y_i|^{q-1} = |x_i|$$

$$= \frac{\sum_{i=1}^{n} |y_i|^q}{\left(\sum_{i=1}^{n} |y_i|^q\right)^{1/p}}$$

$$= \left(\sum_{i=1}^{n} |y_i|^q\right)^{1-\frac{1}{p}} = \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

$$\Rightarrow \qquad \frac{|f(x)|}{1} = \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \|x\|$$

So for particular choice of x, we have

$$\Rightarrow \qquad |f(x)| = \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \|x\| \leq \|f\| \, \|x\|$$

$$\Rightarrow \qquad \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \leq \|f\|$$

Thus necessarily, we have

$$\|f\| = \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \quad \Rightarrow f \in l_q^n \qquad \text{So } x \in l_q^n \qquad \Rightarrow f \in l_q^n$$

Case 2 :- When p = 1, $(l_1^n)^* = l_\infty^n$.

Here we have

$$\|x\| = \sum_{i=1}^{n} |x_i| \qquad \text{where [} x \in l_1^n$$

It follows that

$$|f(x)| = \left|\sum_{1}^{n} x_i y_i\right| \leq \sum_{1}^{n} |x_i y_i|$$

$$= \sum_{1}^{n} |x_i| \, |y_i|$$

$$\leq \max_{1 \leq i \leq n} |y_i| \sum_{1}^{n} |x_i| \qquad \forall\, x = (x_1, \ldots x_n) \in l_1^n.$$

Since we know

$$|f(x)| \le \|f\| \, \|x\|$$

we see that $\|f\| \le \max_{1\le i\le n} |y_i|$

Now $\max_{1\le i\le n} |y_i| = |y_k|$ say for some, k, $1 \le k \le n$.

Choose an $x = (x_1, \ldots, x_n)$

such that $x_i = 0$ if $i \ne k$

$$= x_k = \frac{|y_k|}{y_k}$$

Note that $f \ne 0$, then $\exists\, y_i \ne 0$ such that $y_k \ne 0$.

Thus $|f(x)| = \left|\sum_1^n x_i y_i\right| = \frac{|y_k|.y_k}{y_k} = |y_k|$ by definition of x.

$$\|f\| = \sup_{\|x\|=1} |f(x)| \ge |y_k|$$

since $\left(0,0,\ldots,\frac{|y_k|}{y_k},\ldots.\right)$ has norm 1.

$\Rightarrow$ $\|f\| \ge \max_{1\le i\le n} |y_i|$

$\Rightarrow$ $\|f\| = \max_{1\le i\le n} |y_i|$

So we have $(l_1^n)^* = l_\infty^n$.

Case 3 :- $(l_\infty^n)^* = l_1^n$.

where $\|x\| = \max_{1\le i\le n} |x_i|$

we have $f(x) = \sum_1^n x_i y_i$

$$|f(x)| = \left|\sum_1^n x_i y_i\right| \le \sum_1^n |x_i|\,|y_i|$$

$$= \max_{1\le i\le n} |x_i| \sum_1^n |y_i|$$

Since $|f(x)| \le \|f\|\,\|x\|$

$\Rightarrow$ $\|f\| \le \sum_1^n |y_i|$

consider the vector x defined by

$x_i = 0$ if $y_i = 0$.

$x_i = \dfrac{|y_i|}{y_i}$ otherwise

we have $|f(x)| = \sum_{i=1}^n \dfrac{|y_i| \times y_i}{y_i} = \sum_{i=1}^n |y_i|$

$$\Rightarrow \quad \frac{|f(x)|}{\|x\|} = \frac{\sum_{i=1}^n |y_i|}{\max_{1\le i\le n}\{|x_i|\}} = \frac{\sum_1^n |y_i|}{\max_{1\le i\le n}\left\{\left|\dfrac{|y_i|}{y_i}\right|\right\}}$$

$$= \frac{\sum_1^n |y_i|}{\max_{1\le i\le n} \dfrac{|y_i|}{y_i}} \sum_1^n |y_i|$$

$\Rightarrow$ $|f(x)| = \sum_1^n |y_i|\ \|x\| \le \|f\|\,\|x\|$

$\Rightarrow$ $\sum_1^n |y_i| \le \|f\|$

Thus

$\|f\| = \sum_1^n |y_i|$ where $f \in l_1^n$

Thus $(l_\infty^n)^* = l_1^n$

Remark :- A normed linear space may be complete without being reflexive as we will see

$$(C_0)^* = l_1$$

where C_0 = {space of all convergent sequences converges to zero} and

$$(C_0^*)^* = l_1^* == l_\infty$$

Thus C_0 is not a reflexive. But C_0 is complete space.

Theorem :- C[0, 1] is not regular [reflexive]

Proof :- Here C[0, 1] denotes the set of all real continuous functions

$x = x(t)$ on [0, 1] and

and $$\| x \| = \left(\int_0^1 | x(t) |^2 \, dt \right)^{1/2}$$

Note that C[0, 1] is not a Banach space under this norm.

Assume that C[0, 1] is regular. An arbitrary linear functional F(f) defined on the space V of all functions of bounded variation. Then must have the form $F_x(f) = f(x)$ for suitably chosen $x \in C[0, 1]$. Recalling the general form of functional C[0, 1], we can write for an arbitrary F(f),

$$F_x(f) = f(x) = \int_0^1 x(t)\, df(t) \qquad \ldots(1)$$

where f(t) denotes the function of bounded variation associated with the functional $f(x) \in C[0, 1]$. The functional

$$F_{x_0}(f) = f(t_0 + 0) - f(t_0 - 0) \qquad \ldots(*)$$

assigns to every function f(t) of bounded variation, it jump at the point t_0. Obviously, $F_{x_0}(f)$ is additive and

$$| F_{x_0}(f)| = |f(t_0 + 0) - f(t_0 - 0)|$$

$$\leq \operatorname*{var}_0^1(f) = \|f\|$$

implies the boundedness of $F_{x_0}(f)$ and the fact that norm of $F_{x_0}(f)$ can not be greater than 1. Also $F_{x_0}(f) \neq 0$ that is to say it is sufficient to consider $F_{x_0}(f_1)$ with

$$f_1(t) = \begin{cases} 0 & \text{for } 0 \le t < t_0 \\ t & \text{for } t_0 \le t \le 1 \end{cases}$$

Because of (1), a continuous function $x_0(t)$ can be found such that

$$F_{x_0}(f) = \int_0^1 x_0(t)df(t) \qquad \ldots(2)$$

By (∗) we have

$$F_{x_0}(f_0) = 0$$

for $$f_0(t) = \int_0^1 x_0(t)dt$$

because $f_0(t)$ is continuous on [0, 1]. But on the other hand

$$F_{x_0}(f_0) = \int_0^1 x_0(t)df_0(t) = \int_0^1 x_0(t)dt > 0$$

because $x_0(t) \neq 0$. This is a contradiction. Therefore C[0, 1] can not be regular (reflexive)

Uniform Boundedness Principle

The following theorem i.e. Uniform Boundedness Principle enables us to determine whether the norms of a given collection of bounded linear transformations $\{T_i\}$ have a finite least upper bound or equivalently if there is some uniform bound for the set $(\|T_i\|)$.

So we prove the following results :

Theorem 1: (Banach-Steinhaus or Uniform Boundedness Principle) Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non empty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector in B, then $(\|T_i\|)$ is a bounded set of numbers that is $\{T_i\}$ is bounded as a subset of $\mathcal{B}$(B, N).

Proof : For each positive integer n, let

$$F_n = \{x \,;\, x \in B \text{ and } \| T_i(x) \| \le n \text{ for all } i\}$$

we claim that F_n is a closed subset of B. To show this let y be a limit point of F_n. Then there exists $x \in F_n$ such that $x \neq y$ and $\| x - y \| < \delta$. But since T_i are continuous , we have

$$\| T_i(x) - T_i(y) \| < \in \quad \text{whenever } \| x - y \| < \delta.$$

Now $\quad T_i(y) = T_i(y - x + x)$

and so
$$\| T_i(y) \| = \| T_i(y - x) + T_i(x) \|$$
$$\leq \| T_i(y - x) \| + \| T_i(x) \|$$
$$= \| T_i(y) - T_i(x) \| + \| T_i(x) \|$$
$$< \in + n \quad \text{whenever } \| x - y \| < \delta$$
$$\leq n.$$

Hence $y \in F_n$. Thus F_n is closed. Also by our assumption

$$B = \bigcup_{n=1}^{\infty} F_n$$

Since B is complete , using Baire's Theorem , we see that one of the F_n , say F_{n_i} has non – empty interior and thus contains a closed sphere S_0 with centre x_0 and radius $r_0 > 0$. Therefore each vector in every set $T_i(S_0)$ has norm less than or equal to n_0 , that is $\| T_i(S_0) \| \leq n_0$.

Clearly $S_0 - x_0$ is the closed sphere with radius r_0 centred on the origin and so $\frac{S_0 - x_0}{r_0}$ is the closed unit sphere S. Since x_0 is in S_0 , we have

$$\| T_i(S_0 - x_0) \| = \| T_i(S_0) - T_i(x_0) \|$$
$$\leq \| T_i(S_0 0 \| + \| T_i(x_0) \|$$
$$\leq n_0 + n_0 = 2\, n_0 .$$

This yields

$$\| T_i(S) \| = \| T_i \left(\frac{S_0 - x_0}{r_0} \right) \| \leq \frac{2n_0}{r_0}$$

and therefore

$$\| T_i \| = \sup \{ \| T_i(S) \| ; \quad \| S \| \leq 1 \}$$
$$\leq \sup \left\{ \frac{2n_0}{r_0} \right\}$$

$$= \frac{2n_0}{r_0} \quad \text{for every i.}$$

which completes the proof of the theorem.

Consequences of Uniform Boundedness Principle

We prove some consequence of Banach – Steinhaus Theorem (Uniform Boundedness Principle) having several applications in analysis.

Theorem 2: A non empty subset X of a normed linear space N is bounded if and only if f(X) is a bounded set of numbers for each f in N*.

Proof : Since $|f(x)| \leq \| f \| . \| x \|$, it follows that if X is bounded , then f(X) is also bounded for each f.

To prove the converse , we write $X = \{x_i\}$. We now use natural imbedding to map X to the subset $\{F_{x_i}\}$ of N**. The assumption that $f(X) = \{f(x_i)\}$ is bounded for each f implies that for $\{F_{x_i}(f)\}$ is bounded for each f. Moreover since N* is complete , uniform boundedness theorem shows that $\{F_{x_i}\}$ is a bounded subset of N**. Since natural imbedding preserves norms , therefore X is evidently a bounded subset of N. This completes the proof of the theorem.

Theorem 3: Let X be a Banach space and Y , a normed linear space. Let $\{T_n\}$ be a sequence of terms from ß(X, Y) covering strongly to T. Then there exists a positive constant M such that $\| T_n \| < M$ for all n.

Proof : Since $T_n \xrightarrow{S} T$, then

$$\lim_{n} T_n x = T x \qquad \text{for all x .}$$

This in turn implies that

$$\sup_{n} \| T_n(x) \| < \infty \qquad \text{for all x .}$$

Now using uniform boundedness principle , we must have

$$\sup_{n} \| T_n \| < \infty .$$

and therefore the theorem is proved.

Definition : Let $\{T_n\}$ be a sequence of linear transformation from ß(X , Y). Then $\{T_n\}$ is said to be a strong Cauchy sequence if the sequence $\{T_n(x)\}$ is a Cauchy sequence for all $x \in X$.

Further a space ß(X , Y) is said to be complete in the strong sense if every strong Cauchy sequence in ß(X, Y) converges strongly to some member of the space.

We now prove the following :

Theorem 4: If the spaces X and Y are each Banach spaces , then ß(X , Y) is complete in the strong sense.

Proof : Let $< T_n >$ be a strong Cauchy sequence in ß(X , Y).We must show that there is some element T of ß(X , Y) to which $< T_n >$ converges strongly. Since $< T_n >$ is a strong Cauchy sequence , it follows by definition that for any $x \in X$ $< T_n x >$ is a Cauchy sequence of elements of Y. Since Y is a Banach space, the limit of this sequence must exist in Y. Thus for any $x \in X$, the function

$$Tx = \lim_{n} T_n x \qquad (1)$$

Is defined. Clearly , T is linear transformation and (1) is equivalent to saying that

$$T_n \to T.$$

It remains to show that T is a bounded linear transformation. Since X is a Banach space and $< T_n >$ converges strongly to T, theorem 3 implies that

$$\| T_n \| < M , \quad \text{for all n and some positive constant M.}$$

Since for any $x \in X$, we can say

$$\| T_n x \| \leq \| T_n \| . \| x \|$$

this implies that

$$\| T_n (x) \| \leq M . \| x \|$$

for any x and every n. Since it is true for every n, it must also be true in the limit. Thus

$$\lim_{n} \| T_n(x) \| \leq M . \| x \|.$$

Since norm is continuous , we have

$$\| \lim_{n} T_n x \| \leq M . \| x \|$$

or $\quad \| Tx \| \leq M . \| x \|$

for every x. Hence T is bounded. Thus we have shown that every strong Cauchy sequence in **ß**(X , Y) converges strongly to some element T of ß(X, Y). Hence ß(X , Y) is complete in the strong sense and the proof is complete.

We now define what is meant by a week Cauchy sequence of elements of the normed linear space X.

Definition : The sequence of element $\{x_n\}$ of the normed linear space x is said to be a weak Cauchy sequence if $< f(x_n) >$ is a Cauchy sequence of elements for al $f \in X^*$, the conjugate space of X.

Theorem 5: In a normed linear space X , every Cauchy sequence is bounded.

Proof : Let $< x_n >$ be a weak Cauchy sequence of elements of a normed linear space X. This means that $< f(x_n) >$ is a Cauchy sequence for all $f \in X^*$. We recall the natural imbedding

$$\phi : X \to X^{**}$$

$$x \to F_x$$

where $F_x(f) = f(x)$ for all $x \in X$ and $f \in X^*$. ϕ is a bounded linear functional satisfying

$$\| \phi(x) \| = \| x \| \qquad \text{for all } x \in X.$$

Since $< f(x_n) >$ is a Cauchy sequence of complex numbers , for any $f \in X^*$, we have

$$\sup_n | F_{x_n}(f) | = \sup_n | f(x_n) | < \infty \qquad (1)$$

But X* is a Banach space. Therefore by Uniform Bounded Principle (1) yields

$$\sup_n \| F(x_n) \| < \infty$$

Since $\| F_{x_n} \| = \| \phi(x_n) \| = \| x_n \|$

therefore $\sup_n \| x_n \| < \infty$.

Hence the weak Cauchy sequence $\{x_n\}$ is bounded. This completes the proof.

Theorem 6: In a normed linear space X , if the sequence < x_n > converges weakly to x , that is $x_n \xrightarrow{w} x$, then there exists some positive constant m such that $\| x_n \| < m$ for all n.

Proof : we note that if

$$x_n \xrightarrow{w} x .$$

then certainly < x_n > is a weak Cauchy sequence , Hence by Theorem 5 , { x_n } is bounded , that is $\| x_n \| \leq m$ for constant m and the proof is complete.

After having introduced the definition of weak Cauchy sequence , we give the following definition of weak completeness of a space.

Definition : A normed linear space X is said to be weakly complete if every Cauchy sequence of elements of X converges weak to some other member of X.

Our next theorem shows that any reflexive space is weakly complete.

Theorem 7: If the normed linear space X is reflexive , then it is also weakly complete.

Proof : Suppose < x_n > is a weak Cauchy sequence of elements of X. this means that < $f(x_n)$ > is a Cauchy sequence for all $f \in X^*$. Now we consider natural imbedding

$$\phi : X \to X^{**}$$

$$x \to F_x$$

This mapping implies that < $F_{x_i}(f)$ > is a cauchy sequence of scalars for all $f \in X^*$. Since the underlying field is either real or complex (each of which is complete metric space)

This implies that the functional y defined on X** by

$$y(f) = \lim_{n} F_{x_n}(f)$$

exist for every $f \in X^*$. It can be verified that y is linear. We shall now show that y is a bounded linear functional.

Since $\| F_{x_n} \| = \| x_n \|$ and < x_n > is a Cauchy sequence , it follows by theorem 5 , that there is some positive number M such that

$$\| x_n \| \leq M.$$

for all n, this implies that

$$| F_{x_n}(f) | = | f(x_n) | \leq \| f \| . \| x_n \|$$

$$\leq M . \| f \|$$

for any $f \in X^*$ and all n . Hence it is true in the limit that is

$$\lim | F_{x_n}(f) | \leq M \| f \|$$

$$\Rightarrow \quad | \lim F_{x_n}(f) | \leq M . \| f \|$$

or $$| y(f) | \leq M \| f \| \qquad \text{using (1)}$$

for all $f \in X^*$ and all n.

This however implies that y is a bounded linear functional or that $y \in X^{**}$. Since X is reflexive there must be some $x \in X$ that we can identify with y that is , there must be some $x \in X$ such that $y = F_x$.

Hence for any $f \in X^*$, we can say

$$\lim_{n} f(x_n) = \lim_{n} F_{x_n}(f)$$

$$= y(f)$$

$$= F_x(f)$$

$$= f(x)$$

Since this holds for any $f \in X^*$, we have

$$x_n \xrightarrow{w} x .$$

Thus we have shown that each weak Cauchy sequence of elements of X converges weakly to some other member of X. Hence X is weakly complete and the proof of the theorem is complete.

Open Mapping Theorem and its applications

First we present some definitions which will be required in the sequel.

Definition :- If $T : V \to W$ is a linear transformation, then the set N of all vectors $x \in V$ such that $Tx = 0$ is called the null space (or kernel) of T. Thus

$$N = \{x \in V;\ Tx = 0\} = T^{-1}\{0\}$$

Also $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 \in N$ and that if $x \in N$, then $Tx = 0$ so that if T is injective (one to one). Thus we have shown that T is injective if and only if $N = \{0\}$.

Now suppose that X and Y are normed linear spaces and $T : X \rightarrow X$ is a continuous linear mapping. Let $x_0 \in N$ (null space of T) and let $x_n \rightarrow x$. Since T is continuous $Tx_n \rightarrow Tx$ thus $Tx = \lim Tx_n = \lim 0 = 0$. Hence $x \in N$. This proves that if $T : X \rightarrow Y$ is continuous, then null space of T is closed.

Definition :- Let X and Y be normed linear spaces. Then a linear mapping $T : X \rightarrow Y$ will be called open mapping if it maps open sets into open set.

Definition :- The mapping $T : X \rightarrow Y$ where X and Y are normed spaces as will be called a homeomorphism if it is bijective, continuous and open or equivalently $T : X \rightarrow Y$ is a homeomorphism if it is bijective and bi-continuous.

Definition :- Let E be a normed linear space. A subset A of E is called no-where dense in E if $\overline{A}$ has an empty interior. Q is everywhere dense in R while integers are nowhere dense in R. Thus a nowhere dense set is thought of a set which does not cover much of the space.

Baire Category Theorem :- It states that a complete space can not be covered by any sequence of no-where dense sets.

Open mapping Theorem or Interior Mapping Principle

First of all, we prove a Lemma

Lemma :- Let B and B′ be Banach spaces. If T is a continuous linear transformation of B onto B′, then the image of each open sphere centred on the origin in B contains an open sphere centred on the origin, in B′.

Proof :- Let S_r and S_r' be open spheres with radius r centred on the origin in B and B′ respectively. Then

$$T(S_r) = T(rS_1) = r\,T(S_1)$$

So, it is sufficient to show that $T(S_1)$ contains some S_r'.

We first prove that $\overline{T(S_1)}$ contains some S_r'. Since T is onto, we note that $B' = \bigcup_{n=1}^{\infty} T(S_n)$.

Being a Banach space, B' is complete and so by Baire's theorem, some $\overline{T(S_{n_0})}$ has an interior point y_0 lying in $T(S_{n_o})$. Since the mapping $y \to y - y_0$ is a homeomorphism of B' onto itself. $\overline{T(S_{n_0})} - y_0$ has the origin as an interior point. Since y_0 is in $T(S_{n_0})$ we have

$$T(S_{n_0}) - y_0 \subseteq T(S_{2n0})$$

which in turn implies that

$$\overline{T(S_{n0})} - y_0 = \overline{T(S_{n0}) - y_0} \subseteq \overline{T(S_{2n0})}$$

which shows that the origin is an interior point of $\overline{T(S_{2n0})}$. As we know that multiplication by any non-zero scalar is a homeomorphism of E' onto itself. So

$$\overline{T(S_{2n0})} = \overline{2n_0 T(S_1)} = 2n_0 \overline{T(S_1)}$$

and hence the origin is also an interior point of $\overline{T(S_1)}$. Thus $S'_{\in} \subseteq \overline{T(S_1)}$ for some positive number $\in$. We complete the proof by showing that $S'_{\in} \subseteq \overline{T(S_1)}$ which is equivalent to $S'_{\in/2} \subset T(S_1)$.

Let $y \in B'$ be such that $\|y\| < \in$. Since y is in $\overline{T(S_1)}$, there exists a vector x_1 in B such that $\|x_1\| < 1$ and $\|y - y_1\| < \frac{\in}{2}$, where $y_1 = T(x_1)$. Further $S'_{\in/2} \subset \overline{T(S_{1/2})}$ and $\|y - y_1\| < \frac{\in}{2}$, there exists a vector x_2 in B such that $\|x_2\| < \frac{1}{2}$ and $\|(y - y_1) - y_2\| < \frac{\in}{4}$ where $y_2 = T(x_2)$, continuing in this way, we get a sequence $\langle x_n \rangle$ in B such that $\|x_n\| < \frac{1}{2^{n-1}}$ and

$$\|y - (y_1 + y_2 + \ldots + y_n)\| < \frac{\in}{2^n}$$

where $y_n = T(x_n)$. Let $S_n = x_1 + x_2 + \ldots + x_n$, then

$$\begin{aligned} \|S_n\| &= \|x_1 + x_2 + \ldots + x_n\| \\ &\le \|x_1\| + \|x_2\| + \ldots + \|x_n\| \\ &< 1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}} < 2 \end{aligned}$$

Also for n > m, we have

$$\|S_n - S_m\| = \|x_{m+1} + x_{m+2} + \ldots + x_n\|$$

$$\leq \|x_{m+1}\| + \|x_{m+2}\| + \ldots + \|x_n\|$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \frac{1}{2^{n-1}}$$

$$= \frac{\frac{1}{2^m}\left(1 - \frac{1}{2^{n-m}}\right)}{1 - \frac{1}{2}}$$

$$= \frac{1}{2^{m-1}}\left[1 - \frac{1}{2^{n-m}}\right] \to 0 \text{ as } m, n \to \infty.$$

Hence $\{S_n\}$ is a Cauchy sequence in B and since B is complete, there exists a vector x in B such hat

$$\lim_{n\to\infty} S_n = x \text{ and so}$$

$$\|x\| = \|\lim S_n\| = \lim \|S_n\| \leq 2 < 3$$

which implies that $x \in S_3$. Now

$$y_1 + y_2 + \ldots + y_n = T(x_1) + T(x_2) + \ldots + T(x_n)$$

since T is continuous, $x = \lim S_n$

$$\Rightarrow \quad Tx = \lim_{n} (TS_n)$$

$$= \lim (y_1 + y_2 + \ldots + y_n)$$

$$\Rightarrow \quad Tx = y$$

Thus $y = Tx$ where $\|x\| < 3$ so that $y \in T(S_3)$

Hence we have proved that

$$y \in S'_\in \Rightarrow y \in T(S_3) \text{ and so } S'_\in \subseteq T[S_3]$$

Statement of Open Mapping Theorem.

Let B and B′ be Banach spaces. If T is a continuous linear transformation of B onto B′, then T is an open mapping. (Thus if the mapping T is also one to one, then T^{-1} is continuous).

Proof of Theorem :- It is sufficient to show that if G, is an open set in B, then T(G) is also open in B′. To show it let $y \in T(G)$ we shall show that y is an interior point of T(G) i.e. there exists an open sphere centered on y and contained in T(G). Let x be a point in G such that y = Tx. Since G is open, x is an interior point of G.

Therefore x is the centre of an open sphere written in the form x = Sr, contained in G. Hence by the above Lemma, $T(S_r)$ contains some sphere S'_{r_1}. Then $y + S'_{r_1}$ is an open sphere centred on y.

Moreover $$y + S'_{r_1} \subseteq y + T(S_r)$$

$$= T(x) + T(S_r)$$

$$= T(x + Sr)$$

$$\subseteq T(G)$$

Hence $y + S'_{r_1}$ is an open sphere centred on y and contained in T(G). Consequently T(G) is open. Hence the result.

Theorem :- A one to one continuous linear transformations of one Banach space onto another is a homeomorphism.

Proof :- The given hypothesis yields that the linear transformation is bijective and continuous. Further by open mapping theorem, the linear transformation is also open. Hence it is homeomorphism.

Projections on Banach spaces

Definition :- Let L be a vector space. We say that X is the direct sum of its subspace say M and N; if every element $z \in L$ has a unique representation z = x + y with x in M and y in N. In such a case we write $L = M \oplus N$.

Define a mapping $P : L \to L$ by P(z) = x. Then P is a linear transformation, then

(1) P(z) = z if and only if $z \in M$

(2) P(z) = 0 if and only if $z \in N$

(3) P is idempotent that is $P^2 = P$. Infact

$$P^2(z) = P(P(z)) = P(x) = x = P(z).$$

Such a linear mapping P is called a projection on the linear space L. Thus if L is the direct sum of its subspaces M and N, then there exists a linear transformation P which is idempotent.

But, however in case of Banach spaces, more is required of a projection than more linearity and idempotence we have

Definition :- A projection on a Banach space is a projection on B in the algebraic sense (linear and idempotent) which is also continuous.

It follows from the above discussion that if B is the direct sum of its subspaces M and N, then there exists a linear transformation P which is idempotent. Further we have

Theorem :- If P is a projection on a Banach space B, and if M and N are its range and null space, then M and N are closed linear subspaces of B such that $B = M \oplus N$

Proof :- We are given that P is a projection on a Banach space B and M and N are range and null spaces of P. Thus P is linear, continuous and idempotent and

$$M = \text{range of } P = \{P(z);\, z \in B\}$$

$$N = \text{null space of } P = \{z\, ;\, P(z) = 0\}$$

Let $z \in B$. Consider

$$z = P(z) + (I - P)z \qquad \ldots(1)$$

where I denotes the identity transformation on B such that $I(z) = z$ for all $z \in B$. Clearly P(z) is in M and since P is idempotent, we have

$$\begin{aligned} P\{(I-P)(z)\} &= \{P(I-P)\}(z) \\ &= (P - P^2)(z) \\ &= (P - P)(z) = 0(z) = 0 \end{aligned}$$

It follows therefore that $(I-P)(z) \in N$, the null space of P. Therefore equation (1) gives a de composition of z according to the subspaces M and N. This decomposition is unique because if we have another representation as $z = x + y$, $x \in M$, $y \in N$, then

$$P(z) = P(x) = x$$

and $$(I-P)(z) = I(z) - P(z)$$

$$= z - x$$

$$= y$$

Thus B = M⊕N. We know that the null space of a continuous linear transformation is closed. Therefore continuity of P implies that N is closed. Further, since M = {P(z); z∈B} = {x ;P(x) = x}

$$\Rightarrow \qquad M = \{x ; (I - P)(x) = 0\}$$

It follows that M is the null space of continuous linear transformation I–P and hence closed. Thus M and N are closed and B = M ⊕ N. Hence the result.

As an application of open mapping theorem, we have

Theorem :- Let B be a Banach space and let M and N be closed linear subspaces of B such that B = M ⊕ N. If z = x +y is the unique representation of a vector in B as the sum of vectors in M and N, then the mapping P defined by P(z) = x is a projection on B whose range and null space are M and N.

Proof :- Let P : B→B be defined by P(z) = x for z = x + y, x∈M, y∈N. Then since P(z) = x for z∈B, we have M to be the range of P. Also P(y) = 0 for y∈N. Therefore N is the null space of P.

Further

$$P^2(z) = P(P(z)) = P(x) = x = P(z)$$

Implies that P is idempotent. Hence to prove that P is a projection on B, it only remains to show that P is continuous. Let

$$z = x + y,\ x \in M,\ y \in N$$

be unique representation of the elements of the Banach space B. Define a new norm on B by

$$||z||' = ||x|| + ||y||$$

and let B′ denote the linear space B equipped with this new norm, then B′ is a Banach space and since

$$||P(z)|| = ||x|| \le ||x|| + ||y|| = ||z||'$$

It follows that P in continuous as a mapping of B′ into, B. It is therefore sufficient to show that B and B′ are homeomorphic. Let T denote the identity mapping of B′ onto B. Then

$$||T(z)|| = ||z|| = ||x + y|| \le ||x|| + ||y|| = ||z||'.$$

Shows that T is one to one continuous linear transformation of B′ onto B. Open mapping theorem now implies that T is a homeomorphism. Thus B and B′ are homeomorphic. Hence $P : B \to B$ is continuous and therefore a projection on B.

Closed Linear Transformations and Closed Graph Theorem

Let X and Y be normed linear spaces. Then the Cartesian product $X \times Y$ of X and Y becomes a normed linear space under the norm defined by

$$\|(x, y)\| = \|x\| + \|y\|$$

Further if X and Y are Banach spaces, then $X \times Y$ is also a Banach space w.r.t. the norm defined above.

Definition :- Let $T : B \times B'$ be a linear transformation of a Banach space into another Banach space B′. Then the collection of ordered pairs.

$$G_T = \{(x, Tx); (x, Tx) \in B \times B'\}$$

is called the graph of T. It can be shown that G_T is a linear subspace of $B \times B'$.

Definition :- Let X and Y be normed linear spaces and let D be a subspace of X. Then the linear transformation $T : D \to Y$ is called closed if $\{x_n\} \in D$, $\lim_{n} x_n = x$ and $\lim_{n} Tx_n = y \in Y$ imply $x \in D$ and $y = Tx$.

As justification for the name given closed transformation in the above definition, we now show that a linear transformation T is closed iff its graph G_T is a closed subspace of $X \times Y$.

Theorem A :- A linear transformation is closed iff its graph is a closed subspace.

Proof :- Let X and Y be normed linear spaces and let D be a subspace of X. Suppose first that $T : D \to Y$ is a closed linear transformation. To show that G_T is closed, we must show that any limit point of G_T is actually a member of G_T. Therefore there must be a sequence of points of G_T, (x_n, Tx_n), $x_n \in D$ converging to (x, y), this is equivalent to

$$\|(x_n, Tx_n) - (x, y)\| \to 0$$

or $$\|(x_n - x, Tx_n - y)\| \to 0$$

or $$\|x_n - x\| + \|Tx_n - y\| \to 0$$

$\Rightarrow$ $$x_n \to x \text{ and } Tx_n \to y$$

Since T is closed, this implies that $x \in D$ and $y = Tx$.

Therefore we can write that

$$(x, y) = (x, Tx) \in G_T$$

$\Rightarrow$ Every limit pt (x, y) of G_T is a member of G_T.

$\Rightarrow$ G_T is closed.

Conversely suppose that G_T is closed, and let $x_n \to x$, $x_n \in D$, for all n as $Tx_n \to y$. We must show that $x \in D$ and $y = Tx$. The condition implies that

$$(x_n, Tx_n) \to (x, y) \in \overline{G_T}$$

Since G_T is closed we have

$$G_T = \overline{G}_T \text{ and thus we have.}$$

$$(x, y) \in G_T$$

But by the definition of G_T, this means that $x \in D$ and $y = Tx$. Hence T is a closed linear transformation. This completes the proof of the theorem. The next things we wish to investigate is when a bounded (continuous) transformation is closed. Infact, we prove.

Theorem B :- Let X and Y be normed linear spaces and let D be a closed subspace of X If $T : D \to Y$ is bounded, then T is closed.

Proof :- D is a closed subspace of X and $T : D \to Y$ is bounded. If $\langle x_n \rangle$ is a convergent sequence of points of D such that $Tx_n \to y$, then D being closed, the limit of the sequence $\langle x_n \rangle$ must belong to D. On the other hand, the continuity (boundedness) of T implies that $Tx_n \to Tx$. Hence $y = Tx$. (since $Tx_n \to y$). Thus T becomes closed. Hence the result.

An immediate consequence of the theorem is of the following :

Corollary :- Suppose T is linear transformation from a normed linear space X into another normed linear space Y. If T is continuous, then T is closed. Also then using Theorem A, G_T is closed.

Proof :- We know that the entire space X is always closed, therefore Theorem B applies and the result follows.

Theorem C :- Let X and Y be normed linear spaces and let D be a subspace of X. If $T : D \to Y$ is a closed linear transformation, then T^{-1}(if exists) is also a closed linear transformation.

Proof :- Since T is closed, its graph.

$$G_T = \{(x, Tx);\ x \in D\}$$

is closed, let T(D) denote the range of T. Since T^{-1} exists, for any $y \in T(D)$, there is a unique $x \in D$ such that $y = Tx$ or $x = T^{-1}(y)$. Therefore graph of T can be written as

$$G_T = \{(T^{-1}y;\ y);\ y \in T(D)\}$$

Consider now the mapping

$$X \times Y \to Y \times x$$

$$(x, y) \to (y, x)$$

This mapping is isometry, since Isometrics map closed sets into closed sets and the set $\{(T^{-1}y;\ y)\ y \in T(D)\}$ is closed. It follows that the set $\{(y,\ T^{-1}y)\ y \in T(D)\}$is also closed. But this last set is just the graph of T^{-1}. Thus we have proved that that graph of T^{-1} is closed or hence T^{-1} is closed by Theorem A.

Theorem D :- Let D be a subspace of a normed linear space X and let $T: D \to Y$ be a linear transformation from D into a Banach space Y. If T is closed and bounded, then D is a closed subspace of X.

Proof :- It is sufficient to show that any limit point of D is also a member of D. Hence suppose that x is a limit point of D. This means that there must be some sequence $\{x_n\}$ of points of D such that $x_n \to x$. Consider now

$$\|Tx_n - Tx_m\| \leq \|T\|\ \|x_n - x_m\|$$

Since $\|x_n - x_m\| \to 0$ as $n, m \to \infty$

as every convergent sequence is Cauchy.

It follows that $\langle Tx_n \rangle$ is a Cauchy sequence in Y. But Y being a Banach space is complete. Therefore there exists $y \in Y$ such that

$$Tx_n \to y.$$

Thus we have $x_n \to x$ and $Tx_n \to Y$. Now since T is closed. This implies that $x \in D$. Hence D contains all its limit points and hence closed. This completes the proof of the theorem.

We now state and prove Closed Graph Theorem.

Closed Graph Theorem

Theorem :- Let B and B′ be Banach spaces and let T : $B \to B'$ be a linear transformation. Then graph of T is closed if and only if T is continuous.

Proof :- Suppose first that T is continuous. Then Corollary to Theorem B implies that G_T is closed.

Conversely suppose that G_T is closed. Since B and B′ are Banach spaces. It follows that $B \times B'$ is a Banach space. Since closed subsets of a complete metric space must be complete, it follows that G_T (being closed) is Banach space too. Now consider the mapping

$$f : G_T \to B$$

defined by $$f(x, Tx) = x$$

clearly f is a linear transformation. We claim further that f is bounded. To prove this, we note that

$$\|f(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\|$$

$$= \|(x, Tx)\|$$

which implies that f is a bounded linear transformation. Further $f(G_T) = B$ and therefore f is onto. We shall show that f is one to one. Also we know that a linear transformation is one-to-one if its kernel (null space) consists of identity element only. Therefore. We need to prove that (0, 0) is the only element f maps into zero. Hence, suppose

$$f(x, Tx) = x = 0.$$

But x = 0 implies that Tx = 0 and so

$$(x, Tx) = (0, 0)$$

and hence f is one to one. Thus $f : G_T \to B$ is bijective and therefore f^{-1} exists. Now G_T and B and Banach spaces and f is a continuous linear transformation and f^{-1} is continuous. To complete the proof we must show that if $x_n \to x$, then $Tx_n \to Tx$. [T is continuous]. Hence suppose that $x_n \to x$.

Since f^{-1} is continuous, we have

$$f^{-1}x_n \to f^{-1}x,$$

$$\Rightarrow \quad (x_n, Tx_n) \to (x, Tx)$$

$$\Rightarrow \quad (x_n - x, Tx_n - Tx) \to (0, 0)$$

$$\Rightarrow \quad Tx_n \to Tx$$

Thus T is continuous. Hence the result.

Equivalent Norms

Suppose X is a vector space over the scalar field F and suppose that $\|.\|_1$ and $\|.\|_2$ are each norms on X, Then $\|.\|_1$ is said to be equivalent to $\|.\|_2$ written as $\|.\|_1 \sim \|.\|_2$, if $\exists$ positive numbers a and b such that

$$a \| x \|_1 \leq \| x \|_2 \leq b \| x \|_1 \text{ for all } x \in X.$$

This relation is an equivalence relation on the set of all norms over a given space. Further, if two norms are equivalent, then certainly if $<x_n>$ is a Cauchy sequence with respect to $\|.\|_1$ it must also be a cauchy sequence with respect to $\|.\|_2$ and vice-versa.

Let a basis for he finite dimensional space be $[x_1, x_2, \ldots, x_n]$. For any $x \in X$, there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$. Now $\| x \|_0 = \max_i |\alpha_i|$ is indeed a norm. This norm is called **Zeroth Norm.** We

Theorem :- On a finite dimensional space, all norms are equivalent.

Proof :- We shall show that all norms are equivalent by showing that any norm is equivalent to the particular norm defined above and called the **Zeroth norm.**

Let a basis for the finite dimensional space X is given by

$$x_1, x_2, \ldots, x_n.$$

For any $x \in X_1$ there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i xi \qquad \ldots(*)$$

Now $\qquad \| x \|_0 = \max_i |\alpha_i|$

is indeed a norm.

Now let $\| \ \|$ be any norm on X. We want to find real numbers a, b > 0 such that (1) is satisfied, where $\| \ \|_2$ is replaced by $\| . \|$ and $\| . \|_1$ is replaced by $\| \ \|_0$.

The right hand side of (1) easily satisfies

$$a \| x \|_1 \leq \| x \|_2 \leq b \| x \|_1 \qquad \ldots(1)]$$

since from (*)

$$\| x \| = \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq \sum_{i=1}^{n} |\alpha_i| \, \| x_i \|$$

$$\leq \max_i |\alpha_i| \sum_{i=1}^{n} \| x_i \|$$

$$\leq \| x \|_0 \sum_{i=1}^{n} \| x_i \|$$

because, since the basis is fixed, we can take as the number b

$$b = \sum_{i=1}^{n} \| x_i \|$$

to get for any $x \in X$,

$$\| x \| \leq b \, \| x \|_0$$

The left side of (1) does not follow quite as simply. Consider the simple case of a one-dimensional space with basis x_1. Any vector in the space X can be written uniquely as

$$x = \alpha_1 x_1$$

for some $\alpha_1 \in F$. Hence

$$\| x \| = |\alpha| \, \| x_1 \| = \| x \|_0 \, \| x_1 \|$$

Thus in this case, the number a on the left side of (1) can be taken to be just $\| x_1 \|$. Having verified this, we shall now proceed by induction, suppose the theorem is true for all spaces of dimension less than or equal to $n-1$. We can now say that, if dim X = n, with basis $\{x_1, x_2 \ldots, x_n\}$ and

$$M = [\{x_1, x_2, \ldots, x_{n-1}\}]$$

be the subspace spanned by the first $n-1$ basis vectors, then

$$\| \ \| \sim \| \ \|_0$$

in M. Since this is so, if $\{y_n\}$ is a cauchy sequence of elements from M w.r.t. to $\| \ \|$, then $\{y_n\}$ is also a cauchy sequence with respect to $\| \ \|_0$. Consider the ith term of this sequence now :

$$y_i = \alpha_1^{(i)} x_1 + \alpha_2^{(i)} x_2 + \ldots + \alpha_{n-1}^{(i)} x_{n-1}$$

By the above

$$\| y_n - y_m \|_0 \to 0 \text{ as } n, m \to \infty \qquad \ldots(2)$$

since $\{y_n\}$ is a cauchy sequence.

But $\|y_n - y_m\|_0 = \max_j |\alpha_j^{(n)} - \alpha_j^{(m)}|$

which by (2) implies

$$|\alpha_j^{(n)} - \alpha_j^{(m)}| \to 0 \ \ (\text{as } n, m \to \infty) \qquad \ldots(3)$$

for j = 1, 2,…, n–1. Since F = R or C, and each is complete and (3) states that if the $\{\alpha_j^{(m)}\}$ is a cauchy sequence, there must exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$\alpha_j^{(m)} \to \alpha_j \ (j = 1, 2, \ldots, n-1)$$

In view of this, it is clear that

$$y_m \to y = \sum_{j=1}^{n-1} \alpha_i x_i$$

with respect to the zeroth norm. Further

$$y_m \xrightarrow{\|\,\|_0} y \Rightarrow y_m \xrightarrow{\|\bullet\|} y$$

Thus under the induction hypothesis, are have shown that subspace M is complete with respect to an arbitrary norm which immediately implies that it is closed.

Furthermore, from the above, we see that this statement will be true for any finite dimensional subspace of a normed space. Consider the nth basis vector x_n now and from the set

$$x_n + M = \{x_n + z \mid z \in M\}$$

Since for any y, z $\in$M,

$$\| x_n + z - (x_n + y)\| = \| z - y\|$$

Since x_n + M is seen to be isometric to M under the mapping $z \to x_n + z$. Hence since M is closed, x_n + M must be closed as well which implies that C(x_n + M) is open, [where C(x_n + M) is the complement of x_n + M] we now contend that

$$0 \notin x_n + M$$

for if it did, we would be able to write for some $\beta_1, \beta_2, \ldots, \beta_{n-1} \in F$,

$0 = x_n + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{n-1} x_{n-1}$, which is ridiculous. Also 0 is a point of the open set $C(x_n + M)$; Hence there must be a whole nbd of zero lying entirely within $C(x_n+M)$). In other words, there must exist $C_n > 0$ such that for any

$$x \in x_n + M, \| x-0\| \geq C_n. \;\; 0 \in C(x_n + M)$$

[Note that here we say that the distance from any point $x_n + M$ to zero is positive].

Thus for all $\alpha_i \in F$ $(i = 1, \ldots, n-1)$,

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n-1} x_{n-1} + x_n \| \geq C_n,$$

or
$$\left\| \frac{\alpha_1}{\alpha_n} x_n + \ldots + \frac{\alpha_{n-1}}{\alpha_n} x_{n-1} + x_n \right\| \geq C_n$$

which implies for any $\alpha_n \in F$, that

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq |\alpha_n| \, C_n$$

because we can write for $\alpha_n \neq 0$,

Suppose now that we had not taken

$$M = [\{x_1, x_2, \ldots, x_{n-1}\}]$$

but had taken instead

$$[\{x_1, x_2 \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}]$$

since the only fact about M was that its dimension was $n-1$. It is clear that in an analogous fashion we could have arrived at some $c_i > 0$ such that

$$\| \alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq C_i \, |\alpha_i|$$

for any $i = 1, 2, \ldots, n$. In view of this we can say for any

$$x = \sum_{i=1}^{n} \alpha_i x_i,$$

$$\| \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq \min_i C_i \max_i | \alpha_i | = \min_i C_i \| x \|_0$$

This completes the proof of since $a = \min_i C_i$ is positive.

Corollary 1 :- If X is any finite dimensional normed linear space, X is complete [since all norms are equivalent].

Corollary 2 :- If X is a normed linear space and M is any finite dimensional subspace, M is closed.

Theorem :- Suppose $A : X \to Y$, where X and Y are normed linear spaces. If X is finite dimensional, A is bounded.

Proof :- Suppose dim X = n, that a basis for X is given by

$$x_1, x_2, \ldots, x_n.$$

In view of this for any $x \in X$, scalars $\alpha_1, \alpha_2 \ldots, \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

and A is linear, we have

$$Ax = \sum_{i=1}^{n} \alpha_i Ax_i$$

Letting $K = \sum_{i=1}^{n} \|Ax_i\|$, we have

$$\|Ax\| = \left\| \sum_{i=1}^{n} \alpha_i Ax_i \right\|$$

$$\leq \sum_{i=1}^{n} |\alpha_i| \, \|Ax_i\|$$

$$\leq \|x\|_0 . K.$$

since $\|x\|_0 = \max_i |\alpha_i|$

Since all norms in a finite dimensional space are equivalent and A is bounded with respect to zeroth norm, it follows that A must be a bounded linear transformation no matter what norm is chosen for X.

Weak and Strong convergence

Definition: If $\| T_n - T \| \to 0$, then we say that the sequence $< T_n >$ of operators (or linear transformation) converges to T and this convergence's is called **convergence in norm** or **strong convergence.** The linear transformation T is said to be the strong limit of the sequence $< T_n >$. Also $< T_n >$ is said to converge weakly towards the linear transformation T if the sequence $< T_n(x) >$ converges to Tx.

Definition: Let E be a normed linear space, $< x_n >$ a sequence of elements of E and $x_0 \in E$. if the sequence $f(x_n) \to f(x_0)$ as $n \to \infty$ for all functionals $f \in E^*$, then $< x_n >$ is said to converge weakly to x_0 and we write

$$x_n \xrightarrow{w} x_0.$$

x_0 is called the weak limit of the sequence $< x_n >$.

Remark: A sequence can not converge weakly to two different limits, that is the weak limit of a sequence is unique.

We suppose that $x_n \xrightarrow{w} x_0$ and $x_n \xrightarrow{w} y_0$ i.e $f(x_n) \to f(x_0)$ and $f(x_n) \to f(y_0)$ for an arbitrary linear f. Then

$$f(x_0) = f(y_0) \text{ , or}$$

$$f(x_0 - y_0) = 0$$

Now if we choose an f_0 with $\| f_0 \| = 1$ and $f_0(x_0 - y_0) = \| x_0 - y_0 \|$, then we have

$$\| x_0 - y_0 \| = 0 \text{ i.e. } \quad x_0 = y_0$$

Prop: Let N be a normed linear space and $(x_n) \subseteq N$. Then $x_n \to x$ in norm

implies $\Rightarrow x_n \xrightarrow{w} x$.

Proof: $| f(x_n) - f(x) | = | f(x_n - x) |$

$$\leq \| f \| \| x_n - x \| \to 0 \text{ as } n \to \infty$$

$$[\text{since } x_n \to x \text{ in norm } \forall f \in N^*]$$

$$\Rightarrow x_n \xrightarrow{w} x$$

Remark: Thus by above prop, norm convergence or strong convergence $\Rightarrow$ Weak convergence.

But the weak convergence need not imply strong convergence. However in a finite dimensional normed linear space, the two convergences are equivalent.

Theorem : In a finite dimensional space, the notion of weak and strong convergence are equivalent.

Proof: Since strong convergence $\Rightarrow$ weak convergence always.

For the converse suppose $< x_n >$ converges weakly where i. e. $f(x_n) \to f(x) \ \forall \ f \in E^*$ and E is of finite dimensional. Since E is finite dimensional, $\exists$ a finite system of linearly independent elements $e_1, e_2, \ldots\ldots, e_k$ and every $x \in E$ can be represented in the form

$$x = \xi_1 e_1 + \xi_2 e_2 + \ldots\ldots\ldots\ldots\ldots + \xi_k e_k$$

with real $\xi_1, \xi_2, \ldots\ldots, \xi_k$. Thus

$$x_n = \xi_1^{(n)} e_1 + \xi_2^{(n)} e_2 + \ldots\ldots\ldots\ldots\ldots + \xi_k^{(n)} e_k$$

Now we consider such functionals $f_i \in E^*$ for which $f_i(e_i) = 1$ and $f_i(e_k) = 0$ for $k \neq i$. Thus $f_i(x_n) = \xi_i^{(n)}$ and $f_i(x_0) = \xi_i^{(0)}$

But since the sequence $f(x_n) \to f(x_0)$ for every linear functional f, so also $f_i(x_n) \to f_i(x_0)$ that is

$$\xi_i^{(n)} \to l_{u_i}^{(0)} \quad \text{for } i = 1, 2, \ldots, k$$

Let M be the greatest of the numbers $\|e_i\|$ $(i = 1, 2, \ldots, k)$ i. e. $M = \text{Max} \| e_i \|$. Then for any given
$\in > 0$, $\exists$ an n_0 s. that

$$|\xi_i^{(n)} - \xi_i^{(0)}| < \frac{\in}{M.K}$$

for all $i = 1, 2, \ldots, k$ and $n \geq n_0$. Thus

$$\| x_n - x_0 \| = \| \sum_{i=1}^{k} (\xi_i^{(n)} - \xi_i^{(0)}) e_i \|$$

$$\leq \sum_{i=1}^{k} |(\xi_i^{(n)} - \xi_i^{(0)})| \, \| e_i \|$$

$$< \in.$$

Hence the sequence $< x_n >$ converges strongly to x_0.

Unit-IV

Compact Operations on Normed Spaces

Definition:- Let X and Y be normed spaces. An operator $T : X \to Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if for every bounded subset M of X, the image T(M) is relatively compact that is the closure $\overline{T(M)}$ is compact.

Remark:- Many linear operators in analysis are compact. A systematic theory of compact linear operators emerged from the theory of integral equations of the form

$$(T-\lambda I)\, x(s) = y(s) \text{ where } Tx(s) = \int_a^b K(s, t)\, x(t)dt.$$

where $\lambda \in \mathbb{C}$ is a parameter, y and kernel K are given functions (subject to certain conditions) and x is the unknown function. Such equations also play a role in the theory of ordinary and partial differential equations. The term compact is suggested by the definition. The older term completely continuous can be motivated by the following Lemma which shows that a compact linear operator is continuous but the converse is not generally true.

Relation of Compact and continuous linear operator

Theorem 1. Let X and Y be normed spaces. Then

(a) Every compact linear operator $T : X \to Y$ is bounded, hence continuous

(b) If $\dim X = \infty$, the identity operator $I : X \to X$ (which is continuous) is not compact.

Proof (a) Since the unit sphere $U = \{x \in X : \| x \| = 1\}$ is bounded and T is compact, so by definition $\overline{T(U)}$ is compact. Now since every normed space is metric space and by the result "Every compact subset of a metric space is closed and bounded." so that

$$\sup_{\|x\|=1} \| Tx \| < \infty.$$

Hence T is bounded. But by the result "Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces. Then

(1) T is continuous if and only if T is bounded.

(2) If T is continuous at a single point, T is continuous".

Thus T is continuous. Hence every compact linear operator $T : X \to Y$ is bounded and hence continuous.

(b) Since the closed unit ball $M = \{x \in X;\ \|x\| \leq 1\}$ is bounded. If $\dim X = \infty$, then by the result "If a normed space X has the property that the closed unit ball $M = \{x\ ;\ \|x\| \leq 1\}$ is compact, then X is finite dimensional" M can not be compact. Thus $I(M) = M = \overline{M}$ is not relatively compact.

Remark :- From the definition the compactness of a set, we obtain a useful criterion for operators.

Theorem 2 :- Let X and Y be normed spaces and $T : X \to Y$ be linear operator. Then T is compact if and only if it maps every bounded sequence $<x_n>$ in X onto a sequence $<Tx_n>$ in Y which has a convergent subsequence.

Proof :- If T is compact and $<x_n>$ is bounded, then the closure of $<Tx_n>$ in Y is compact. Since every normed space is metric space and by the definition, "a metric space X is said to be compact if every sequence in X has a convergent subsequence". Thus $<Tx_n>$ contains a convergent subsequence.

Conversely assume that every bounded sequence $<x_n>$ contains a subsequence $< x_{n_k} >$ such that $< Tx_{n_k} >$ converges in Y. Consider any bounded subset $B \subset X$, and let $<y_n>$ be any sequence in T(B). Then $y_n = Tx_n$ for some $x_n \in B$ and $<x_n>$ is bounded since B is bounded. But by assumption $<Tx_n>$ contains a convergent subsequence. Hence by definition of compactness, $\overline{T(B)}$ is compact. Since $<y_n>$ in T(B) was arbitrary. Thus by definition of compact operator, T is compact.

Remark :- The sum $T_1 + T_2$ of two compact linear operators from normed space X to normed space Y is compact. Similarly αT_1 is compact, where α is any scalar. Thus the compact linear operators from X into Y form a vector space.

Compactness of linear transformation on a finite dimensional space

Theorem 3 :- Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then

(a) If T is bounded and $\dim T(X) < \infty$, the operator T is compact.

(b) If dim X $<\infty$, the operator T is compact.

Proof (a) :- Let $<x_n>$ be any bounded sequence in X. Then the inequality $\| Tx_n\| \leq \|T\| \cdot \|x_n\|$ shows that $<Tx_n>$ is bounded. Now by the result " In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded" and dim (X) $< \infty$ implies that $<Tx_n>$ is relatively compact. It follows that $<Tx_n>$ has a convergent subsequence. But by Theorem 2, $T : X \to Y$ is compact if and only if T maps every bounded sequence $<x_n>$ in X onto a sequence $<Tx_n>$ in Y which has a convergent subsequence". Hence the operator T is compact.

(b) Since we know that if a normed space X is finite dimensional then every linear operator on X is bounded operator. Thus T is bounded. Also dim $X < \infty$. Now by the result "If T is a linear operator and dim D(T) = $n<\infty$, then dim R(T) $\leq$ n "where D(T) and R(T) are domain and range of T." Thus if dim (X) $< \infty$, then dim T(X) $< \infty$. Now since dim T(X) $< \infty$ and T is bounded. It follows by (a) part that the operator T is compact.

Compactness of the limit of the sequence of Compact Operators

Theorem 4 :- Let $< T_n >$ be a sequence of compact linear operators from a normed space X into a Banach space Y. If $< T_n >$ is uniformly operator convergent, say $\|T_n - T\| \to 0$, then the limit operator T is compact.

Proof :- Using a diagonal method, we show that for any bounded sequence $<x_m>$ in X, the image $<Tx_m>$ has a convergent subsequence and then apply Theorem 2 i.e. "Let X and Y be normed spaces and $T : X \to Y$, a linear operator. Then T is compact if and only if it maps every bounded sequence $<x_n>$ in X onto a sequence $<Tx_n>$ in Y which has a convergent subsequence."

Since T_1 is compact, $<x_m>$ has a subsequence $<x_{1,m}>$ such that $<T_1 x_{1,m}>$ is Cauchy. Similarly $<x_{1,m}>$ has a subsequence $<x_{2,m}>$ such that $<T_2\, x_{2,m}>$ is Cauchy. Continuing in this way, we see that the diagonal sequence $<y_m> = <x_{m,m}>$ is a subsequence of $<x_m>$ such that for every fixed positive integer n, the sequence $<T_n\, y_m>_{m\in N}$ is Cauchy. $<x_m>$ is bounded, say $\|x_m\| \leq c$ for all m. Hence $\|y_m\| \leq c$ for all m. Let $\in>0$. Since

$$T_m \to T, \text{ there is an } n = p \text{ such that } \|T-T_p\| < \in/3c \qquad \ldots(1)$$

Since $<T_p\, y_m>_{m\in N}$ is Cauchy, there is an N such that

$$\|T_p\, y_j - T_p\, y_k\| < \frac{\in}{3} \qquad \ldots(2)$$

$$(j, k > N)$$

Hence we obtain for j, K > N

$$\|Ty_j - Ty_k\| \leq \|Ty_j - T_p y_j\| + \|T_p y_j - T_p y_k\| + \|T_p y_k - Ty_k\|$$

$$\leq \|T - T_p\| \,.\, \|y_j\| + \frac{\in}{3} + \|T_p - T\| \,.\, \|y\|$$

$$< \frac{\in}{3c}\,.\,c + \frac{\in}{3} + \frac{\in}{3c}\,.\,c \qquad \text{(Using (1) and (2))}$$

$$= \in$$

This shows that $<Ty_m>$ is cauchy and converges since Y is complete. But $<y_n>$ is a subsequence of the arbitrary bounded sequence $<x_m>$. Hence using theorem 2, which states that "Let X and Y be normed spaces and $T : X \to Y$, a linear operator. Then T is compact if and only if it maps every bounded sequence $<x_n>$ in X onto a sequence $<Tx_n>$ in Y which has a convergent subsequence," we get that the operator T is compact.

Remark :- The above theorem states conditions under which to limit of a sequence of compact linear operators is compact. This theorem is also important as a tool for proving compactness of a given operator by exhibiting it as the uniform operator limit of a sequence of compact linear operators.

Note that the present theorem becomes false if we replace uniform operator convergence by strong operator convergence $\|T_n x - Tx\| \to 0$. This can be seen from $T_n : l^2 \to l^2$ defined by $T_n(x) = (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$

Where $x = (\xi_i) \in l^2$. Since T_n is linear and bounded, T_n is compact by Theorem 3(a). Clearly $T_n x \to x = Ix$ but I is not compact since $\dim l^2 = \infty$.

The following example illustrates how the theorem can be used to prove compactness of an operator.

Example (space l^2). To prove compactness of $T : l^2 \to l^2$ defined by $y = (\eta_j) = Tx$ where $\eta_j = \xi_j/j$ for $j = 1, 2, \ldots.$

Solution :- T is linear. If $x = (\xi_j) \in l^2$, then $y = (\eta_j) \in l^2$. Let $T_n : l^2 \to l^2$ be defined by

$$T_n x = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots, \frac{\xi_n}{n}, 0, 0, \ldots.\right)$$

T_n is linear and bounded and is compact by Theorem 3(a), Further

$$\|(T - T_n)x\|^2 = \sum_{j=n+1}^{\infty} |\eta_j|^2 = \sum_{j=n+1}^{\infty} \frac{1}{j^2} \,.\, |\xi_j|^2$$

$$\leq \frac{1}{(n+1)^2} \sum_{j=n+1}^{\infty} |\xi_j|^2 \leq \frac{\|x\|^2}{(n+1)^2}$$

Taking the supremum over all x of norm 1, we see that

$$\|T - T_n\| \leq \frac{1}{n+1}.$$

Hence $T_n \to T$ and hence T is compact by the above Theorem 4.

Theorem 5 :- Let X and Y be normed spaces and $T : X \to Y$ a compact linear operator. Suppose that $<x_n>$ in X is weakly convergent, say, $x_n \xrightarrow{w} x$. Then $<Tx_n>$ is strongly convergent in Y and has the limit y = Tx.

Proof :- We write $y_n = Tx_n$ and $y = Tx$. First we show that

$$y_n \xrightarrow{w} y. \qquad \ldots(1)$$

Then we show that

$$y_n \to y \qquad \ldots(2)$$

Let g be any bounded linear functional on Y. We define a functional f on X by setting

$$f(z) = g(Tz) \qquad (z \in X)$$

f is linear, f is bounded because T is compact, hence bounded and

$$|f(z)| = |g(Tz)| \leq \|g\| \cdot \|Tz\| \leq \|g\| \cdot \|T\| \cdot \|z\|$$

By definition $x_n \xrightarrow{w} x$ implies $f(x_n) \to f(x)$, hence by the definition, g $(Tx_n) \to g(Tx)$, that is, $g(y_n) \to g(y)$ since g was arbitrary, this implies that $y_n \xrightarrow{w} y$ which proves (1).

Now we prove (2). Assume that (2) does not hold. Then $<y_n>$ has a subsequence $<y_{n_k}>$ such that

$$\|y_{n_k} - y\| \geq \eta \qquad \ldots(3)$$

for some $\eta > 0$. Since $<x_n>$ is weakly convergent, by the result "Let $<x_n>$ be a weakly convergent sequence in a normed space X, say $x_n \xrightarrow{w} x$, then the sequence $(\|x_n\|)$ is bounded". Thus $<x_n>$ is bounded and so is $<x_{n_k}>$. But by Theorem 2, "Let X and Y be normed spaces and $T : X \to Y$, a linear operator. Then T is compact if and only if it maps every bounded sequence $<x_n>$ in X onto a sequence $<Tx_n>$ in Y which has a convergent subsequence", since the operator T is compact, $< Tx_{n_k} >$ has a convergent subsequence say $< \bar{y}_j >$. Let $\bar{y}_j \to \bar{y}$. Hence $\bar{y}_j \xrightarrow{w} \bar{y}$. Since by the result " Let $<x_n>$ be a weakly convergent sequence in a normed space X, say $x_n \xrightarrow{w} x$, then every subsequence of $<x_n>$ converges weakly to x", Thus by this result and (1) we have $\bar{y}$ = y. consequently

$$\| \bar{y} - \bar{y} \| \to 0$$

But $\| \bar{y}_i - y \| \geq \eta > 0$ [By (3)]

This contradicts, so that (2) must hold.

Closed Range Theorem

Definition:- Suppose X is a Banach space, M is a subspace of X and N is a subspace of X* (Dual space of X) , neither M nor N is assumed to be closed. Their annihilators $M^{\perp}$ and $N^{\perp}$ are defined as follows:

$$M^{\perp} = \{x^* \in X^*, < x, x^*> = 0 \quad \text{for all } x \in M\}$$
$$N^{\perp} = \{x \in X , < x, x^*> = 0 \quad \text{for all } x^* \in N\}$$

Thus $M^{\perp}$ consists of all bounded linear functionals on X that vanish on M and $N^{\perp}$ is the subset of X on which every member of N vanishes. It is clear that $M^{\perp}$ and $N^{\perp}$ are vector spaces. Since $M^{\perp}$ is the intersection of the null spaces of the functionals, $M^{\perp}$ is a weak* closed subspace of X*.

The weak*-topology of X* is by definition , the weakest one that makes all functionals

$$x^* \to < x, x^* >$$

continuous. Thus the norm topology of X* is stronger than its weak*-topology.

Notation:- If T maps X into Y, then the null space of T and range of T will be denoted by N(T) and $\Re(T)$ respectively

$$N(T) = \{x \in X , T x = 0\}$$

$$\Re(T) = \{ y \in Y ; T x = y \text{ for some } x \in X\}.$$

Theorem :- If X and Y are Banach spaces and if $T \in$ **B**(X , Y) [set of bounded or continuous linear operator] , then each of the following three conditions implies the other two:

(a)$\Re(T)$ is closed in Y.

(b) $\Re(T^*)$ is weak*-closed in X*.

(c) $\Re(T^*)$ is norm-closed in X^*.

Proof: It is obvious that (b) implies (c). We will prove that (a) implies (b) and that (c) implies (a).

Suppose (a) holds. Then $N(T)^{\perp}$ is the weak closure of $R(T^*)$.

To prove (b), it is therefore enough to show that

$$N(T)^{\perp} \subset \Re(T^*)$$

Pick $x^* \in N(T)^{\perp}$. Define a linear functional Λ on $\Re(T)$ by

$$\Lambda\, T\, x = < x , x^* > \qquad (x \in X)$$

Note that Λ is well defined for if $T\,x = T\,x'$, then $x - x' \in N(T)$, hence

$$< x - x' , x^* > = 0$$

The open mapping theorem applies to

$$T : X \to \Re(T)$$

since $\Re(T)$ is assumed to be a closed subspace of the complete space Y and is therefore complete. It follows that there exists $K < \infty$ such that to each $y \in \Re(T)$ corresponds an $x \in X$ with $T\,x = y$, $\| x \| \leq K \| y \|$ and

$$| \Lambda\, y | = | \Lambda\, T\, x | = | < x , x^* > | \leq K \| y \| . \| x^* \|$$

Thus Λ is continuous. By the Hahn-Banach theorem some $y^* \in Y^*$ extends Λ. Hence

$$< Tx , y^* > = \Lambda\, T\, x = < x , x^* > \quad (x \in X)$$

This implies $x^* = T^* y^*$. Since x^* was an arbitrary element of $N(T)^{\perp}$, we have shown that

$$N(T)^{\perp} \subset \Re(T^*)$$

Thus (b) follows from (a).

Suppose next that (c) holds. Let **Z** be the closure of $\Re(T)$ in Y. Define some S $\in$ **B**(X , **Z**) by setting $Sx = Tx$. Since $\Re(S)$ is dense in **Z**.

Thus $\qquad S^* : \mathbf{Z}^* \to X^*$

is one-to-one.

If $z^* \in \mathbf{Z}^*$, then by Hahn-extensions theorem, we get an extension y^* of z^*, for every $x \in X$,

$$< x , \mathbf{T}^* y^* > = < \mathbf{T}x , y^* > < Sx , z^* > = < x , \mathbf{S}^* \mathbf{z}^* >$$

Hence $\mathbf{S}^* z^* = T^* y^*$. It follows that S^* and T^* have identical ranges. Since (c) is assumed to hold. $\Re(S^*)$ is closed , hence complete.

Apply the open mapping theorem to

$$\mathbf{S^*} : \mathbf{Z^*} \to \Re(S^*)$$

Since $\mathbf{S}^*$ is one to one , the conclusion is that there is a constant $c > 0$ which satisfies

$$c \| z^* \| \leq \| S^* z^* \|$$

for every $z^* \in \mathbf{Z^*}$.

Now using the following result

"Suppose U and V are the open unit balls in the Banach space X and Y , respectively. Suppose $T \in \mathbf{B}(X , Y)$ and $C > 0$,

(a) If the closure of T(U) contains $c\mathbf{V}$, then

$$T(U) \supset c\mathbf{V}$$

(b) If $c \| y^* \| \leq \| T^* y^* \|$ for every $y^* \in Y^*$, then

$$T(U) \supset c\mathbf{V}."$$

We have , $S : X \to \mathbf{Z}$ is an open mapping , in particular $S(X) = Z$.
But $\Re(T) = \Re(S)$, by the definition of S.

Thus $\Re(T) = Z$, a closed subspace of $\mathbf{Y}$.

This completes the proof that (c) implies (a).

Definition: An inner product space X or pre – Hilbert space is a complex linear space together with an inner product $(,) : X \otimes X \to C$ such that

(i) $(x, y) = \overline{(y, x)}$ [complex conjugate of (y, x)]

(ii) $(\lambda x + \mu y, z) = \lambda(x , z) + \mu(y, z)$

(iii) $(x , x) \geq 0$ and $(x, x) = 0$ iff $x = 0$

condition (i) clearly reduces to $(x, y) = (y, x)$ if X is real vector space. From (i) and (ii), we obtain

$$(x, c\, y + d\, z) = \overline{(cy + dz, x)}$$

$$= \bar{c}\,\overline{(y,x)} + \bar{d}\,\overline{(z,x)}$$

$$= \bar{c}\,(x, y) + \bar{d}\,(x, z)$$

In any pre-Hilbert space, the following are immediate

(a) (x, y + z) = (x, y) + (x, z)

(b) (x, λy) = $\bar{\lambda}$ (x,y)

(c) (0, y) = (x, 0) = 0

(d) (x–y, z) = (x, z) – (y, z)

Examples

1. Let C^n be the vector space of n tuples. If x = (λ_1, λ_2,….., λ_n) and y = (μ_1,….., μ_n) define

$$(x, y) = \sum_{k=1}^{n} \lambda_k \bar{\mu}_k$$

Then all the axioms for pre – Hilbert space are satisfied. This example is known as n – dimensional unitary space and will be denoted by C^n. In this space, the norm of x is defined by

$$\| x \| = \left(\sum_{i=1}^{n} | \lambda_i |^2 \right)^{1/2}$$

2. Let C(a, b) be the vector space of continuous functions defined on [a, b], a < b. Define

$$(x, y) = \int_a^b x(t).\ \overline{y(t)}\, dt$$

With respect to this inner product, C[a, b] is a pre- Hilbert space. The norm of x in C[a, b] is introduced by taking

$$\| x \| = \left(\int_a^b | x(t) |^2 \, dt \right)^{1/2}$$

3. Let P be the vector space of finitely non – zero sequences. If x = (λ_k) and y = (μ_k), define

$$(x, y) = \sum_{k=1}^{\infty} \lambda_k \bar{\mu}_k$$

This space is a pre-Hilbert space with respect to this inner product. The norm of x in this space is defined by

$$\| x \| = \left(\sum_{k=1}^{\infty} | \lambda_k |^2 \right)^{1/2}$$

Theorem 1 : Each Inner Product space is a normed linear space under $\| x \| = (x, x)^{1/2}$. Since all the properties of norm are satisfied. We notice that

(i) $\| x \| = (x, x)^{1/2} \geq 0$

(ii) $\| x \| = 0 \Leftrightarrow (x, x) = 0 \qquad \text{iff } x = 0$

(iii) $\| \alpha x \|^2 = (\alpha x, \alpha x)$

$$= \alpha \bar{\alpha} (x, x)$$

$$= | \alpha |^2 \| x \|^2$$

$\Rightarrow \| \alpha x \| = | \alpha | \| x \|$

(iv) For $x, y \in X$, we have

$$\| x + y \|^2 = (x + y, x + y) \equiv (x, x + y) + (y, x + y)$$

$$= (x, x) + (y, x) + (x, y) + (y, y)$$

$$= (x, x) + (y, y) + (x, y) + \overline{(x, y)}$$

$$= (x, x) + (y, y) + 2 R(x, y)$$

$$\leq \| x \|^2 + \| y \|^2 + 2 \| x \| \| y \|$$

$$= (\| x \| + \| y \|)^2$$

$$\Rightarrow \| x + y \| \leq \| x \| + \| y \|$$

Therefore, each pre-Hilbert space is a normed linear space.

Theorem 2 : The Inner product (Scalar Product) is a continuous function with respect to norm convergence. (**Inner Product in an Hilbert space is jointly continuous**)

Proof: If $x_n \to x$ and $y_n \to y$, then the number $\| x_n \|$, $\| y_n \|$ are bounded. Let M be their upper bound. Then

$$| (x_n, y_n) - (x, y) | = | (x_n, y_n) - (x_n, y) + (x_n, y) - (x, y) |$$

$$\leq | (x_n, y_n) - (x_n, y) | + | (x_n, y) - (x, y) |$$

$$= | (x_n, y_n - y) | + | (x_n - x, y) |$$

$\leq \| x_n \| \| y_n - y \| + \| x_n - x \| \| y \|$ (By Schwarz inequality)

$\leq M \| y_n - y \| + \| y \| \| x_n - x \|$

Now since $\| y_n - y \| \to 0$ and $\| x_n - x \| \to 0$ as $n \to \infty$, therefore

$$| (x_n, y_n) - (x, y) | \to 0 \text{ for } n \to \infty \text{ and hence } (x_n, y_n) \to (x, y)$$

Thus inner product in a pre-Hilbert space is jointly continuous.

Theorem 3 (Cauchy - Schwarz Inequality): If x and y are any two vectors in an inner product space, then

$$| (x, y) | \leq \| x \| \| y \|$$

Proof: We have

$$(x + \lambda y, x + \lambda y) \geq 0 \quad \text{for arbitrary complex } \lambda.$$

$$\Rightarrow (x, x + \lambda y) + \lambda(y, x + \lambda y) \geq 0$$

$$\Rightarrow (x, x) + \bar{\lambda} (x, y) + \lambda[(y, x) + \bar{\lambda} (y, y)] \geq 0.$$

$$\Rightarrow (x, x) + \bar{\lambda} (x, y) + \lambda(y, x) + \lambda \bar{\lambda} (y, y) > 0$$

if we put is $\lambda = \frac{-(x, y)}{(y, y)}$, then

$$(x, x) - \frac{\overline{(x, y)}(x, y)}{(y, y)} - \frac{(x, y)(y, x)}{(y, y)} + \frac{(x, y)}{(y, y)} \frac{\overline{(x, y)}\,(y, y)}{\overline{(y, y)}} \geq 0$$

$$\Rightarrow (x, x) - \frac{|(x, y)|^2}{(y, y)} \frac{(x, y)(y, x)}{(y, y)} + \frac{(x, y)(y, x)}{(y, y)} \geq 0$$

$$\Rightarrow (x, x) - \frac{|(x, y)|^2}{(y, y)} \geq 0$$

$$\Rightarrow | (x, y) |^2 \leq (x, x)\,(y, y)$$

$$= \| x \|^2 . \| y \|^2$$

$$\Rightarrow | (x, y) | \leq \| x \| \| y \|$$

Theorem 4 (Parallelogram Law): In an Hilbert space H,

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2 \qquad \forall\ x, y \in H.$$

Proof: Writing out the expression on the left in terms of inner products.

$$\| x + y \|^2 + \| x - y \|^2 = (x + y, x + y) + (x - y, x - y)$$

$$= (x, x) + (x, y) + (y, x) + (y, y) + (x, x) - (x, y) - (y, x) + (y, y)$$

$$= 2(x, x) + 2(y, y)$$

$$= 2 \| x \|^2 + 2 \| y \|^2$$

Polarization Identity

Theorem 5 : In a pre – Hilbert space, (inner – product space)

$$(x, y) = \frac{1}{4} [\| x + y \|^2 - \| x - y \|^2 + i \|x + i y\|^2 - i \| x - i y \|^2]$$

Proof: we note that

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2 + (x, y) + (y, x) \qquad (1)$$

Replace y by –y, iy by –iy and obtain

$$\| x - y \|^2 = \| x \|^2 + \| y \|^2 - (x, y) - (y, x)$$

and

$$\| x + iy \|^2 = \| x \|^2 + \| y \|^2 - i(x, y) + i(y, x)$$

$$\| x - iy \|^2 = \| x \|^2 + \| y \|^2 + i(x, y) - i(y, x)$$

It follows that

(2) $- \| x - y \|^2 = - \| x \|^2 - \| y \|^2 + (x, y) + (y, x)$

(3) $i\| x + iy \|^2 = i\| x \|^2 + i\| y \|^2 + (x, y) - (y, x)$

(4) $-i\| x - iy \|^2 = -i\| x \|^2 - i\| y \|^2 + (x, y) - (y, x)$

Adding (1), (2), (3) and (4), we get

$$\| x + y \|^2 - \| x - y \|^2 + i \| x + i y \|^2 - i \| x - i y \|^2 = 4 (x, y)$$

This completes the proof.

Definition: A complete pre – Hilbert space (Inner Product space) is called Hilbert space. Thus a Banach space whose norm is generated by inner product is called Hilbert space.

Example: Denote by H, the set of all sequences $x = (\lambda_k)$ of complex number such that

$$\sum_{k=1}^{\infty} |\lambda_k|^2 < \infty$$

If $x = (\lambda_k)$ and $y = (\mu_k)$ are sequences belonging to H, then by the parallelogram law for complex numbers,

$$|\lambda_k + \mu_k|^2 + |\lambda_k - \mu_k|^2 = 2|\lambda_k|^2 + 2|\mu_k|^2$$

Hence

$$\sum_{k=1}^{n} |\lambda_k + \mu_k|^2 \le 2\sum_{k=1}^{n} |\lambda_k|^2 + 2\sum_{k=1}^{n} |\mu_k|^2$$

for all n. Hence $\sum_{k=1}^{n} |\lambda_k + \mu_k|^2 < \infty$ by the comparison test. Hence the sequence $(\lambda_k + \mu_k)$ belongs to H , that is $x + y \in H$. Furthermore if $x = (\lambda_k)$ belongs to H and λ is a complex number, then $\sum_{k=1}^{n} |\lambda\lambda_k|^2 = |\lambda|^2 \sum_{k=1}^{n} |\lambda_k|^2$ shows that the sequence $(\lambda \lambda_k)$ is absolutely summable, it is denoted by λ x. With respect to the operations x + y and λx , H becomes a linear space. We also note that if $x = (\lambda_k)$ and $y = (\mu_k)$ belong to H, then the series

$$\sum_{k=1}^{\infty} \lambda_k \overline{\mu_k}$$

converges absolutely. In fact, a and b are real numbers, $(a - b)^2 \ge 0$ leads to $ab \le \frac{1}{2}(a^2 + b^2)$ and in particular, we have

$$|\lambda_k \overline{\mu_k}| \le \frac{1}{2} (|\lambda_k|^2 + |\overline{\mu_k}|^2$$

Thus $\sum_{k=1}^{\infty} |\lambda_k \overline{\mu_k}|$ converges by the comparison test.

This justifies the definition of the inner product for H as

$$(x, y) = \sum_{k=1}^{\infty} \lambda_k \overline{\mu_k}$$

The axioms for a pre – Hilbert space are easily verified. The norm of an element x in this space is defined by

$$\| x \| = \left(\sum_{k=1}^{\infty} | \lambda_k |^2 \right)^{\frac{1}{2}}$$

It can be seen that

$$\| \lambda x \| = | \lambda | . \| x \|$$

and that

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$$

Thus to prove that H is a Hilbert space, it is sufficient to show that H is complete.

Suppose $x_1, x_2, \ldots$, is a Cauchy sequence in H, that is $\| x_m - x_n \| \to 0$ as m, n $\to \infty$, say $x_n = (\lambda_k^n)$

For each k, $|\lambda_k^m - \lambda_k^n| \leq \sum_{j=1}^{\infty} |\lambda_j^m - \lambda_j^n|^2 = \| x_m - x_n \|^2$

shows that the sequence $\lambda_k^1, \lambda_k^2, \ldots$, of k th components is Cauchy. Since the complex numbers are complete, $\lambda_k^n \to \lambda_k$ as $n \to \infty$ for suitable λ_k. It will be shown that $\sum_{k=1}^{\infty} |\lambda_k|^2 < \infty$ and that $< x_n >$ converges to $x = (\lambda_k)$.

Let $\in > 0$ be given. Let p be an index such that $\| x_m - x_n \|^2 \leq \in$ whenever m, n $\geq$ p. Fix any positive integer r, then we have

$$\sum_{k=1}^{r} |\lambda_k^m - \lambda_k^n|^2 \leq \| x_m - x_n \|^2 \leq \in$$

provided m, n $\geq$ p. Letting $m \to \infty$,

$$\sum_{k=1}^{r} |\lambda_k - \lambda_k^n|^2 \leq \in$$

provided $n \geq p$, since r is arbitrary, we get

$$\sum_{k=1}^{\infty} |\lambda_k - \lambda_k^n|^2 \leq \in \qquad \text{whenever } n \geq p \qquad (1)$$

In particular, $\sum_{k=1}^{\infty} | \lambda_k - \lambda_k^p |^2 \leq \in$

Hence the sequence $< \lambda_k - \lambda_k^p >$ belongs to H. Adding to it, the sequence $< \lambda_k^p >$ of H, we obtain $(\lambda_k) = x$ belongs to H. It follows from (1) that $\| x - x_n \|^2 \leq \in$ whenever $n \geq p$. Thus $x_n \to x$ and hence H is complete. This Hilbert space of absolutely square summable sequences is denoted by l^2.

Theorem 6 : In a pre – Hilbert space, every cauchy sequence is bounded.

Proof: Let $< x_n >$ be a cauchy sequence and let N be an index such that $\| x_n - x_m \| \leq 1$ whenever $m, n \geq N$. If $n \geq N$, then

$$\| x_n \| = \| (x - x_N) + x_N \|$$
$$\leq \| x - x_N \| + \| x_N \|$$
$$\leq 1 + \| x_N \|$$

Thus if M is the largest of the numbers $1 + \| x_N \|, \| x_1 \|, \ldots\ldots, \| x_{N-1} \|$, we have $\| x_n \| \leq M$ for all n. Hence $< x_n >$ is bounded.

Theorem 7: In any pre – Hilbert space, if $< x_n >$ and $< y_n >$ are Cauchy sequence of vectors, then
$\{(< x_n, y_n >)\}$ is Cauchy (hence convergent) sequence of scalars.

Proof: By Cauchy – Schwarz inequality

$$| (x_n, y_n) - (x_m, y_m) | = | (x_n - x_m, y_n - y_m) + (x_m, y_n - y_m) + (x_n - x_m, y_m) |$$

$$\leq | (x_n - x_m , y_n - y_m) | + | (x_m, y_n - y_m) | + | (x_n - x_m, y_m) |$$

$$\leq \| (x_n - x_m \| . \| y_n - y_m \| + \| x_m \| . \| y_n - y_m \| + \| x_n - x_m \| . \| y_m \|$$

for all m and n. Since $\| x_m \|$ and $\| y_m \|$ are bounded. Therefore by the above theorem, R. H. S. of the above inequality $\to 0$ ad m, $n \to \infty$. Therefore $\{(x_n, y_n)\}$ is cauchy sequence of scalars and hence convergent.

Remark: It follows from this theorem, that in a pre – Hilbert space if $< x_n >$ is a Cauchy sequence, then (x_n, x_n) and hence $\| x_n \|$ is a cauchy sequence of scalars, and hence convergent.

It is clear from the definition that every Hilbert space is a Banach space. We shall see that converse need not be true. The question arises under what condition, a Banach space will become a Hilbert space. In this direction, we have the following result.

Theorem 8 : A Banach space is a Hilbert space $\Leftrightarrow$ ||gm (parallelogram) law holds.

Proof: Let H be a Hilbert space. Thus it is by definition, a Banach space whose norm arises from the inner product taken as $\| x \| = (x, x)^{1/2}$

Then

$$\| x + y \|^2 + \| x - y \|^2 = (x + y, x + y) + (x - y, x - y)$$
$$= (x, x) + (y, y) + (x, y) + (y, x) + (x, x)$$
$$+ (y, y) - (x, y) - (y, x)$$
$$= 2(x, x) + 2(y, y)$$
$$= 2 \| x \|^2 + \| y \|^2.$$

Thus if H is a Hilbert space, then it is a Banach space satisfying || gm law.

Conversely suppose that H is a Banach space and that in H, ||gm law holds good.

We define an inner product in H by

$$(x, y) = \frac{1}{4} [\| x + y \|^2 - \| x - y \|^2] \qquad (1)$$

Then $(x, x) \geq 0$ and $(x, x) = 0 \Leftrightarrow x = 0$ Moreover $(x, x) = \| x \|^2$ and $(x, y) = (y, x)$.

It is only to show that

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y)$$

and

$$(\alpha x, y) = \alpha(x, y)$$

by ||gm law, we note that

$$\| u + v + w \|^2 + \| u + v - w \|^2 = 2 \| u + v \|^2 + 2 \| w \|^2$$

and

$$\| u - v + w \|^2 + \| u - v - w \|^2 = 2 \| u - v \|^2 + 2 \| w \|^2.$$

so that on substracting.

$$\| u + v + w \|^2 + \| u + v - w \|^2 - \| u - v + w \|^2 - \| u - v - w \|^2$$

$$= 2 \| u + v \|^2 - 2 \| u - v \|^2.$$

$$\Rightarrow (u + w, v) + (u - w, v) = 2(u, v) \qquad \text{[using (1)]}$$

$$= (2u, v) \qquad (2)$$

Setting u = w, this implies

$$(2u, v) = 2(u, v)$$

Now let $x_1 = u + w$, $x_2 = u - w$ and $y = v$ to obtain.

$$(x_1, y) + (x_2, y) = (x_1 + x_2, y) \quad [\text{using } (2)]$$

Similarly

$$(a\, x, y) = a(x, y)$$

Thus a Banach space satisfying ||gm is a Hilbert space.

Example of a Banach space which is not Hilbert space

Example 1: We know that a Banach space is a Hilbert space if and only if ||gm Law holds.

Consider the linear space L_1 [0, 1] consisting of equivalence classes of functions summable on [0, 1] w.r.t. to Lebesgue measure with the norm of f $\in L_1[0, 1]$ as

$$\| f \| = \int_0^1 | f(x) |\, dx \qquad (1)$$

$L_1[0, 1]$ is a Banach space under this norm.

We show that this norm does not satisfy || law and thus precludes the possibility of viewing this space as a Hilbert space.

Consider the sets $A = [0, \frac{1}{2}]$ and $B = [\frac{1}{2}, 1]$ and the characteristic functions of these sets χ_A and χ_B. We note that (1) yields.

$$\| \chi_A + \chi_B \|^2 = (\int_0^1 | \chi_A + \chi_B |)^2$$

$$= (\int_0^{1/2} | \chi_A + \chi_B | + \int_{1/2}^1 | \chi_A + \chi_B |)^2$$

$$= \left[\frac{1}{2}+\frac{1}{2}\right]^2 = 1^2 = 1$$

$$\| \chi_A + \chi_B \| = (\int_0^1 | \chi_A + \chi_B |)^2)^{1/2}$$

$$= \left(\int_0^{1/2} |\chi_A + \chi_B| + \int_{1/2}^{1} |\chi_A - \chi_B| \right)^2$$

$$= \left(\frac{1}{2}+\frac{1}{2}\right)^2 = 1$$

But $$2\|\chi_A\|^2 + |\chi_B\|^2 = 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Thus

$$\|\chi_A + \chi_B\|^2 + \|\chi_A - \chi_B\|^2 \neq 2\|\chi_A\|^2 + 2\|\chi_B\|^2.$$

and therefore ||gm Law is not satisfied and hence $L_1[0, 1]$ is not a Hilbert space.

Convex Sets

Definition: A convex set in a Banach space. B is a non empty subset S such that x, y $\in$ S $\Rightarrow$ x(1 – t) + t y $\in$ S for every real number t satisfying $0 \leq t \leq 1$.

If we put $t = \frac{1}{2}$, we see that

$$x, y \in S \Rightarrow \frac{x+y}{2} \in S.$$

Theorem 9 : A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Proof: We know that being convex C is non empty and x, y $\in$ C $\Rightarrow \frac{x+y}{2} \in$ C.

Let d = Inf { $\|x\|$, x $\in$ C }. There exists a sequence $\{x_n\}$ of vectors such that $\|x_n\| \to d$. By the convexity of C, $\frac{x_n + x_n}{2}$ is in C. $\left\|\frac{x_n + x_n}{2}\right\| \geq d$ so $\|x_n + x_n\| \geq 2d$. By ||gm Law, we have

$$\|x_m + x_n\|^2 + \|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2$$

$$\Rightarrow \|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2$$

$$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2$$

$$\to 2d^2 + 2d^2 - 4d^2 = 0 \quad [\|x_n\| \to d] \text{ as } m, n \to \infty .$$

Therefore $\{x_n\}$ is a Cauchy sequence in C. Since H is complete and C is closed; C is complete and their exists a vector x in C such that $x_n \to x$. It is clear by the fact that

$$\|x\| = \| \lim x_n \| = \lim \| x_n \| = d$$

that x is a vector in C with smallest norm. To see that x is unique, suppose that x′ is a vector in C other than x which also has norm d. Then $\frac{x + x'}{2}$ is also in C and we have by ||gm law

$$\| \frac{x + x'}{2} \|^2 = \frac{\| x \|^2}{2} + \frac{\| x' \|^2}{2} - \| \frac{x - x'}{2} \|^2$$

$$< \frac{\| x \|^2}{2} + \frac{\| x' \|^2}{2} = d^2$$

which contradicts the definition of d.

Orthogonal Complements

Definition: Two vectors x and y in a Hilbert space H are said to be orthogonal written

$$(x \perp y) \text{ if } (x, y) = 0$$

Since $\overline{(x, y)} = (y, x)$ we have

$x \perp y \Leftrightarrow y \perp x$. It is also clear that $x \perp 0$ for every x. Moreover since $(x, x) = \| x \|^2$, 0 is the only vector orthogonal to itself,

if $x \perp y$, then

$$\| x + y \|^2 = \| x - y \|^2 = \| x \|^2 + \| y \|^2$$

(This is known as Pythagorean theorem).

Definition: A vector x is said to be orthogonal to a non empty set S (written as $x \perp S$) if $x \perp y$ for every $y \in S$.

Definition: The set of all vectors orthogonal to S is called orthogonal complement of S and is denoted by $S^{\perp}$.

Theorem 10 : Let M be a closed linear subspace a Hilbert space H, let $x \notin M$, and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that

$$\| x - y_0 \| = d.$$

Proof: Let M be a closed linear subspace of H, $x \notin M$ and d be the distance from x to M. Then

$$d = \text{Inf } \{ \| x - y \|; y \in M \}$$

Select a sequence $\{y_n\}$ in M such that $\lim\limits_{n \to \infty} \| x_n - y_n \| = d$. Then by parallelogram law

$$\| y_m - y_n \|^2 = \| (y_m - x) - (y_n - x) \|^2 .$$

$$= 2 \| y_m - x \|^2 + 2 \| y_n - x \|^2$$

$$- \|(y_m - x) + (y_n - x) \|^{2.}$$

$$= 2 \| y_m - x \|^2 + 2 \| y_n - x \|^2 - \| y_m + y_n - 2x \|^2$$

$$= 2\| y_m - x \|^2 + 2 \| y_n - x \|^2 - 4 \left\| \frac{y_m + y_n}{2} - x \right\|^2 .$$

Since $\frac{y_m + y_n}{2} \in M$, we have

$$\left\| \frac{y_m + y_n}{2} - x \right\| \geq d.$$

Therefore

$$\| y_m - y_n \|^2 \leq 2 \| y_m - x \|^2 + 2 \| y_n - x \|^2 - 4d^2$$

$$\to 2d^2 + 2d^2 - 4d^2 = 0, m, n \to \infty.$$

Hence $\{y_n\}$ is a Cauchy sequence in a closed linear space of a complete space H.

Therefore $\exists$ an element $y_0 \in M$ such that

$$y_0 = \lim_{n \to \infty} y_n. \text{ Also}$$

$$d = \lim_{n \to \infty} \| x - y_n \| = \| x - \lim y_n \|$$

$$= \| x - y_0 \|$$

Uniqueness of y_0 : Suppose y_1 and y_2 are two vectors in M s. that $\| x - y_1 \| = d$ and $\| x - y_2 \| = d$. Then to show that $y_1 = y_2$. Since M is a subspace of H, therefore

$$y_1, y_2 \in M \Rightarrow \frac{(y_1 + y_2)}{2} \in M.$$

Hence by the definition of d, we have

$$\| x - \frac{y_1 + y_2}{2} \| \geq d \text{ so that } \| 2x - (y_1 + y_2) \| \geq 2d.$$

By parallelogram Law, we have

$$\|(x-y_1) - (x-y_2)\|^2 = 2\|x-y_1\|^2 + 2\|x-y_2\|^2 - \|(x-y_1) + (x-y_2)\|^2$$

$$\Rightarrow \| y_2 - y_1 \|^2 = 2 \| x - y_1\|^2 + 2 \| x - y_2 \|^2 - \| 2x - (y_1 + y_2) \|^2$$

$$\leq 2d^2 + 2d^2 - 4d^2 = 0$$

Thus $\| y_2 - y_1\|^2 \leq 0$. But $\| y_2 - y_1\|^2 \geq 0$

$$\Rightarrow \| y_2 - y_1 \| = 0 \Rightarrow y_2 - y_1 = 0 \Rightarrow y_1 = y_2.$$

Theorem: If M is a proper closed linear subspace of a Hilbert space H, then there exists a non zero vector z_0 in H such that $z_0 \perp M$.

Proof: Since M is a proper linear subspace of H, then there is a vector x in H which does not belong to M. Let d be distance from x to M. Then (by the above theorem) there exists a vector y_0 in M such that

$$\| x - y_0 \| = d.$$

Define $z_0 = x - y_0$

Since $d > 0$, z_0 is a non – zero vector, we shall show that $z_0 \perp M$. It is sufficient to show that if y is an arbitrary vector in M
Then $z_0 \perp y$.

For any scalar α, we have

$$\| z_0 - \alpha y \| = \| x - (y_0 + \alpha y) \| \geq d = \| z_0 \|$$

$$\Rightarrow \| z_0 - \alpha y \|^2 - \| z_0 \|^2 \geq 0$$

$$\Rightarrow (z_0 - \alpha y, z_0 - \alpha y) - \| z_0 \|^2 \geq 0$$

$$\Rightarrow (z_0, z_0) - \bar{\alpha}(z_0, y) - \alpha(y, z_0) + \alpha\bar{\alpha}(y, y) - \| z_0 \|^2 \geq 0$$

$$\Rightarrow \| z_0 \|^2 - \bar{\alpha}(z_0, y) - \alpha(y, z_0) + | \alpha |^2 \| y \|^2 - \| z_0 \|^2 \geq 0$$

$$\Rightarrow - \bar{\alpha}(z_0, y) - \alpha(y, z_0) + | \alpha |^2 \| y \|^2 \geq 0 \qquad (1)$$

Set $\alpha = \beta (z_0, y)$ for an arbitrary real number β. Then (1) becomes

$$- 2 \beta | (z_0, y) |^2 + \beta^2 | (z_0, y) |^2 \| y \|^2 \geq 0.$$

If we now put $a = | (z_0, y) |^2$ and

$b = \| y \|^2$, we obtain

$$- 2 \beta a + \beta^2 a b \geq 0$$

i.e. $$\beta a(\beta b - 2) \geq 0 \qquad (2)$$

for all real β. However if $a > 0$, then (2) is obviously false for all sufficient small +ve β. We see from this that $a = 0$ i.e. $(z_0, y) = 0$ which implies that $z_0 \perp y$ Hence the theorem.

Theorem 12 : If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M + N$ is also closed.

Proof: Let z be a limit point of M + N. It suffices to show that $z \in M + N$. Let $< z_n >$ be a sequence of points in M + N such that $z_n \to z$. By the assumption that $M \perp N$, we see that M and N are disjoint, so each z_n can be written uniquely in the form $z_n = x_n + y_n$, where $x_n \in M$ and $y_n \in N$. For each $\in > 0$, there exists a +ve integer N such that

$$\| z_m - z_n \| < \in \; \forall \; m, n \geq N(\in)$$

$$\Rightarrow \| z_m - z_n \|^2 < \in^2$$

$$\Rightarrow \| (x_m + y_m - (x_n + y_n) \|^2 < \in^2$$

$$\Rightarrow \| (x_m - x_n) + (y_m - y_n) \|^2 < \in^2$$

$$\Rightarrow \| x_m - x_n \|^2 + \| y_m - y_n \|^2 < \in^2$$

$$\Rightarrow \| x_m - x_n \| < \in, \qquad \| y_m - y_n \| < \in.$$

Thus $< x_n >$ and $< y_n >$ are Cauchy sequences.

But M and N are closed linear subspaces of H and therefore, complete. Hence there exists vectors x and y in M and N respectively such that

$$x_n \to x \text{ and } y_n \to y \text{ . Then}$$

$$z = \lim z_n = \lim (x_n + y_n) = \lim x_n + \lim y_n = x+y \in M + N \text{ .}$$

Thus every limit point of M + N is in M + N and hence M + N is also closed.

Projection Theorem

Theorem 13. If M is a closed linear subspace of a Hilbert space H, then

$$H = M \oplus M^{\perp} \text{ where } M^{\perp} = \text{ The set of all vectors orthogonal to M.}$$

Proof. Since M and $M^{\perp}$ are orthogonal closed linear subspaces of H, by the Previous – Theorem, $M + M^{\perp}$ is also a closed linear subspace of H . Moreover, since $M \perp M^{\perp}$, we have

$M \cap M^{\perp} = \{0\}$. So it is sufficient to show that $H = M + M^{\perp}$. If this is not so, then $M + M^{\perp}$ is a proper closed linear subspace of H and therefore $\exists$ a vector $z_0 \neq 0$ such that $z_0 \perp (M + M^{\perp})$ which is possible only when $z_0 \perp M$ and $z_0 \perp (M + M^{\perp})$ that is when $z_0 \in M^{\perp}$ and $z_0 \in M^{\perp\perp}$ that is when $z_0 \in M^{\perp} \cap M^{\perp\perp}$. But this is impossible since $M^{\perp} \cap M^{\perp\perp} = \{0\}$. Hence $H = M + M^{\perp}$.

UNIT – V

ORTHONORMAL SETS

Definition: A non empty subset $\{e_1, e_2, \ldots, e_n, \ldots\}$ of H is called **orthonormal** if.

$$(e_i, e_j) = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \qquad \text{Kronoecker Delta } \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Thus orthonormal set consists of mutually orthogonal unit vectors [$\| e_i \| = 1$ for every i].

If H contains only the zero vector, then it has no orthonormal sets. If H contains a non – zero vector x and if we normalize x by considering $e = \frac{x}{\| x \|}$, then the single element set {e} is clearly an orthonormal set. In general if $\{x_i\}$ is a non empty set of orthogonal non – zero vector in H and if x_i's are normalized by replacing each of them by $e_i = \frac{x_i}{\| x_i \|}$,Then the resulting set $\{e_i\}$ is an orthonormal set. If should be noted that if $< x_i >$ is a non – empty set of mutually orthogonal non – zero vectors in H and if in this set, each x_i is replaced by the corresponding unit vector $e_i = \frac{x_i}{\| x_i \|}$, then the resulting set $\{e_i\}$ is an orthonormal set.

Example 1: The subset $\{e_1, e_2, \ldots, e_n\}$ of l_2^n where e_i is the n- tuple with 1 in the ith place and 0's elsewhere, then $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set in this space.

Example 2: If $\{e_n\}$ is a sequence with 1 in the nth place, and zero elsewhere, then $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set in l_2.

Theorem 1 : Let $\{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H, then

$$\sum_{i=1}^{n} | (x, e_i) |^2 \leq \| x \|^2 \qquad (1)$$

and further

$$x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j \qquad (2)$$

Proof: The inequality (1) follows from the following computation.

$$0 \le \| x - \sum_{i=1}^{n} (x, e_i)\, e_i \|^2$$

$$= (x - \sum_{i=1}^{n} (x, e_i)\, e_i ,\quad x - \sum_{j=1}^{n} (x, e_j)\, e_j)$$

$$= (x, x - \sum_{j=1}^{n} (x, e_j)\, e_j) - \sum_{i=1}^{n} (x, e_i)\, (e_i , x) - \sum_{j=1}^{n} (e_i, e_j)\, e_j)$$

$$= (x, x - \sum_{j=1}^{n} \overline{(x, e_j)} (x, e_j) - \sum_{i=1}^{n} (x, e_i)\, [(e_i , x - \sum_{j=1}^{n} \overline{(x, e_j)}\, (x, e_j)]$$

$$= (x, x) - \sum_{j=1}^{n} \overline{(x, e_j)}\, (x, e_j) - \sum_{i=1}^{n} (x, e_i)\, (e_i, x)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i)\overline{(x, e_j)}\, (e_i, e_j)$$

$$= \| x \|^2 - \sum_{i=1}^{n} (x, e_i)\, \overline{(x, e_i)} - \sum_{j=1}^{n} (x, e_j)\, \overline{(x, e_j)}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i)\overline{(x, e_j)}\, (e_i, e_j)$$

$$= \| x \|^2 - \sum_{i=1}^{n} | (x, e_i) |^2$$

$$\Rightarrow \sum_{i=1}^{n} | (x, e_i) |^2 \le \| x \|^2$$

Also we observe that

$$(x - \sum_{i=1}^{n} (x, e_i)\, e_i, e_j) = (x, e_j) - \sum_{i=1}^{n} (x, e_i)\, (e_i, e_j)$$

$$= (x, e_j) - (x, e_j)$$

$$= 0\ .$$

Hence

$$x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j \text{ for each } j .$$

Inequality (1) is called the special case of a more general inequality known as Bessel's Inequality.

Theorem 2 : If $< e_i >$ is an orthonormal set in a Hilbert space H and if x is any vector in H, then the set $S = \{ e_i ; (x, e_i) \neq 0 \}$ is either empty or countable.

Proof: For each positive integer n, consider the set

$$S_n = \{e_i ; |(x, e_i)|^2 > \frac{\| x \|^2}{n} \}$$

S_n can not contain more than n – 1 vectors, since in that case $\sum_{i=1}^{p} |(x, e_i)|^2 > \| x \|^2$ when p > (n – 1) and thus contradicts the above theorem. Also, each member of S is contained in $\bigcup_{n=1}^{\infty} S_n$. But union of a countable collection of countable sets is countable. Therefore $\bigcup_{n=1}^{\infty} S_n$ and hence S is countable.

Bessel's Inequality

Theorem 3 : If $< e_i >$ is an orthonormal set in a Hilbert space H, then

$$\sum |(x, e_i)|^2 \leq \| x \|^2$$

for every vector $x \in H$.

Proof: Let $S = \{ e_i, (x, e_i) \neq 0 \}$. If S is empty, then we define $\sum |(x, e_i)|^2$ to be the number zero and the result is obvious in this case. We now assume that S is non – empty. Then by the above theorem, it must be finite or countably infinite. If S is finite, then it can be written in the form

$$S = \{ e_1, e_2, \ldots, e_n\}$$

for some +ve integer n. In this case, we define $\sum |(x, e_i)|^2$ to be $\sum_{i=1}^{n} |(x, e_i)|^2$. The inequality to be proved now reduces to

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x \|^2$$

which has already been proved.

Now consider the case

$$S = [e_i, (x, e_i) \neq 0]$$

is countably infinite.

Let the vectors in S be arranged in a definite order.

$$S = [e_1, e_2, \ldots\ldots, e_n, \ldots\ldots]$$

By the theory of absolutely convergent series, if $\sum_{n=1}^{\infty} |(x, e_n)|^2$ converges, then every series obtained from it by rearranging its terms and also converges and all such series have the same sum. We, therefore, define $\Sigma\ |(x, e_i)|^2$ to be $\sum_1^{\infty} |(x, e_n)|^2$ and it follows from the above remark that $\sum_{n=1}^{\infty} |(x, e_n)|^2$ is a non – negative extended real number which depends only on S and not on the arrangement of its vectors. We now observe that

$$\sum |(x, e_i)|^2 = \sum_{i=1}^{\infty} |(x, e_i)|^2$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} |(x, e_i)|^2$$

$$\leq \lim_{n \to \infty} \|x\|^2 = \|x\|^2$$

Hence

$$\sum |(x, e_i)|^2 \leq \|x\|^2 \text{ for every } x \in H.$$

Theorem 4 : If $< e_i >$ is an orthonormal set in a Hilbert space H, and if x is any vector in H, then

$$x - \sum (x, e_i)\, e_i \perp e_j$$

For each j.

Proof: we set

$$S = \{e_i, (x, e_i) \neq 0 \}$$

when S is empty, we define $\sum$ $(x, e_i)\, e_i$ to be the vector zero and then the required result reduces to the statement that $x - 0 = x$ is orthogonal to each e_j, which is precisely, what is meant by saying that S is empty.

When S is non – empty and finite, then it can be written in the form.

$$S = < e_1, e_2, \ldots, e_n >$$

and we define $\sum$ $(x, e_i)\, e_i$ to be $\sum_{i=1}^{n}$ $(x, e_i)\, e_i$ and in that case the required result reduces to $x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j$ which has already been proved.

We may assume for the remainder of proof that S is countably infinite. Let the vectors in S be listed in a definite order $S = < e_1, e_2, \ldots, e_n, \ldots >$. We put $S_n = \sum_{i=1}^{n} (x, e_i)\, e_i$ and we note that for $m > n$, we have

$$\| S_m - S_n \|^2 = \| \sum_{i=n+1}^{m} (x, e_i)\, e_i \|^2 = \sum_{i=n+1}^{m} |(x, e_i)|^2 \le \| x \|^2.$$

Bessel's inequality shows that the series $\sum_{n=1}^{\infty} |(x, e_n)|^2$ converges and so $< S_n >$ is a Cauchy in H and since H is complete, this sequence converges to a vector S, which we write in the form $S = \sum_{n=1}^{\infty} (x, e_n)\, e_n$.

We now define $\sum$ $(x, e_i)\, e_i$ to be $\sum_{n=1}^{\infty} (x, e_n)\, e_n$ (without considering the effect of rearrangement) and observe that the required result follows from $x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j$ and the continuing of the inner product.

$$\begin{aligned}(x - \sum (x, e_i)\, e_i, e_j) &= (x - S, e_j) \\ &= (x, e_j) - (S, e_j) \\ &= (x, e_j) - (\lim S_n, e_j) \\ &= (x, e_j) - \lim (S_n, e_j) \\ &= (x, e_j) - (x, e_j) = 0.\end{aligned}$$

All that remains to show that this definition of $\sum$ (x, e_i) e_i is valid in the sense that it does not depend on the arrangement of vectors in S. Let the vectors in S be rearranged in any manner;

$$S = \{ f_1, f_2, \ldots\ldots, f_n, \ldots.\}$$

We put $S_n' = \sum_{i=1}^{n} (x, f_i)\, f_i$ and we see as above that the sequence $< f_n >$ converges to the limit S′, which we write in the form $S' = \sum_{n=1}^{\infty} (x, f_n)\, f_n$. We conclude the proof by showing that S′ equals S. Let $\in > 0$ be given and let n_0 be +ve integer so large that if $n \geq n_0$, then $\| S_n - S \| < \in$, and $\| S_n' - S' \| < \in$ and $\sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \in^2$. For some +ve integer $m_0 > n_0$, all terms of S_{n_0} occur among those of S'_{m_0}, so $S'_{m_0} - S'_{n_0}$ is a finite sum of terms of the form (x, e_i) e_i for $e = n_0 + 1, n_0 + 2, \ldots.$ This yields $\| S'_{m_0} - S_{n_0} \|^2 \leq \sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \in^2$ so $\| S'_{m_0} - S'_{n_0} \| < \in$ and

$$\| S' - S \| \leq \| S' - S'_{m_0} \| + \| S'_{m_0} - S_{n_0} \| + \| S_{n_0} - S \| < \in + \in + \in = 3 \in$$

Since $\in$ is arbitrary, this shows that S′ = S.

Definition: An orthonormal set E = {e_i} in a Hilbert space H is said to be complete if the only vector orthogonal to all elements of E is zero. Thus an orthonormal set $< e_i >$ is complete if there does not exist a single vector which is orthogonal to all vectors in E, unless the vector is zero. That is, if it is not possible to adjoin a vector e to $< e_i >$ in such a way that $< e_i, e >$ is an orthonormal set which properly contains $< e_i >$.

Theorem 5 : Every non – zero Hilbert space contains a complete orthonormal set.

Proof: Let H be a non – zero Hilbert space and $x \in H$, $x \neq 0$. Normalize x by writing $e = \frac{x}{\| x \|}$, then clearly $< e >$ is an orthonormal set. It follows therefore that every non – zero Hilbert space surely contains orthonormal sets. Consider the collection of all possible orthonormal sets in H, then the collection has a maximal member M since by Zorn's lemma, if P is partially ordered set in which every chain has an upper bound, then P possesses a maximal element, we shall show that M is complete. Suppose that $y \neq 0$ and $y \perp M$ then put

$$z = \frac{y}{\| y \|}$$

we observe that M $\cup$ < z > is also an orthonormal set and thus contradicts the maximality of M. Hence y $\perp$ M only if y = 0.

Theorem 6: Let H be a Hilbert space and let < e_i > be an orthonormal set in H. Then the following conditions are all equivalent to one another:

(1) < e_i > is complete

(2) x $\perp$ < e_i > $\Rightarrow$ x = 0.

(3) If x is any arbitrary vector in H, then x = $\sum$ (x, e_i) e_i.

(4) If x is any arbitrary vector in H, then $\| x \|^2 = \sum | (x, e_i) |^2$

Proof: (1) $\Rightarrow$ (2), Let < e_i > be complete, if (2) is not zero, then $\exists$ a vector x $\neq$ 0, such that x $\perp$ < e_i >. Define e = $\frac{x}{\| x \|}$, the vector e(is a unit vector and) is then orthogonal to each member of < e_i > . Hence the set obtained by joining e to < e_i > becomes an orthonormal set containing < e_i > {e, e_i = 0} becomes an orthonormal set containing < e_i >. This contradicts the completeness of < e_i >. Hence x $\perp$ < e_i > $\Rightarrow$ x = 0 .

(2) $\Rightarrow$ (3). Suppose that x $\perp$ < e_i > $\Rightarrow$ x = 0. Let x be an arbitrary element in H, then x – $\sum$ (x, e_i) e_i is orthogonal to each e_j for all j and therefore to < e_i >. Therefore (2) implies that x–$\sum$ (x, e_i)e_i = 0

$$\Rightarrow \qquad x = \sum (x, e_i)\, e_i$$

(3) $\Rightarrow$ (4). Suppose that x is an arbitrary vector in H such that x = $\sum$ (x, e_i) e_i. Then by inner product, we have

$$\| x \|^2 = (x, x) = \left(\sum (x, e_i)\, e_i, \sum (x, e_j)\, e_j\right)$$

$$= \sum (x, e_i) \left\{ \sum \overline{(x, e_j)} \right\} (e_i, e_j)$$

$$= \sum (x, e_i)\, \overline{(x, e_i)}$$

$$= \sum | (x, e_i) |^2.$$

(4) $\Rightarrow$ (1). We are given that if x is an arbitrary vector in H, then

$\| x \|^2 = \sum | (x, e_i) |^2$. Suppose that $< e_i >$ is not complete, then it is a proper subset of an orthonormal set $< e_i, e >$. Since e is orthogonal to all e_i's such that $\| e \| = 1$, we have

$$\| e \|^2 = \sum | (e, e_i) |^2 = 0$$

$$\Rightarrow e = 0$$

this contradicts the fact that e is a unit vector. Hence $< e_i >$ is complete.

Remark: If $< e_i >$ is a complete orthonormal set in a Hilbert space H and let x be an arbitrary vector in H, then the numbers $< x , e_i >$ are called Fourier coefficients of x, the expression $x = \sum (x, e_i) e_i$ is called the Fourier expansion of x and equation $\| x \|^2 = \sum | (x, e_i) |^2$ is called **Parseval's equation**.

Example: Consider the Hilbert space $L_2(0, 2\pi)$. This space consists of all complex functions defined on $[0, 2\pi]$ which are Lebesgue measurable and square integrable in the sense that $\int_0^{2\pi} | f(x) |^2 dx < \infty$.

Norm and Inner product in $L_2(0, 2\pi)$ are defined by

$$\| f \| = \left(\int_0^{2\pi} |f(x) |^2 dx \right)^{1/2}$$

$$(f, g) = \int_0^{2\pi} f(x) . \overline{g(x)} dx$$

A simple computation shows that the function e^{inx} for $n = 0, \pm 1, \pm 2, \ldots.$ are mutually orthogonal in L_2.

$$\int_0^{2\pi} e^{imx} e^{-inx} dx = \begin{cases} 0, & m \neq n \\ 2\pi & if\ m = n \end{cases}$$

It follows from this that the functions e_n $(n = 0, \pm 1, \pm 2, \ldots..)$ defined by $e_n(x) = e^{inx} / \sqrt{2\pi}$ from an orthonormal set in L_2. For any function f in L_2, the numbers

$$C_n = (f, e_n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, e^{-inx}\, dx \qquad (1)$$

are its classical Fourier coefficients and Bessel's inequality takes the form.

$$\sum_{n=-\infty}^{\infty} |C_n|^2 \le \int_0^{2\pi} |f(x)|^2\, dx. < \infty.$$

It is a fact of very great importance in the theory of Fourier series that the orthonormal set $< e_n >$ is complete in L_2. As we have seen that for every f in L_2, Bessel's inequality can be strengthened to Parseval's equation :

$$\sum_{n=-\infty}^{\infty} |C_n|^2 = \int_0^{2\pi} |f(x)|^2\, dx.$$

The previous theorem also tells us that the completeness of $< e_n >$ is equivalent to the statement that each f in L_2 has a Fourier expansion

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} C_n e^{inx}.$$

Gram – Schmide Orthogonalization Process

Suppose that $< x_1, x_2 \ldots\ldots, x_n, \ldots >$ is a linearly independent set in a Hilbert space H. Our aim is to convert it into the corresponding orthonormal set $< e_1, e_2, \ldots\ldots, e_n, \ldots >$ with the property that for each n, the linear subspace of H is spanned by $< e_1, e_2, \ldots\ldots, e_n, \ldots >$

Our first step is to normalize x_1 by putting

$$e_1 = \frac{x_1}{\| x_1 \|}$$

Let us consider $x_2 - (x_2, e_1) e_1$. It is orthogonal to e_1 and we normalize this by putting

$$e_2 = \frac{x_2 - (x_2, e_1)e_1}{\| x_2 - (x_2, e_1)e_1 \|}$$

Now e_1 and e_2 are orthogonal. Consider $x_3 - (x_3, e_1)\, e_1 - (x_3, e_2)\, e_2$. It is orthogonal to e_1 and e_2. We normalize it by

$$e_2 = \frac{x_3 - (x_3, e_1)e_1 - (x_3, e_2)e_2}{\| x_3 - (x_3, e_1)e_1 - (x_3, e_2)e_2 \|}$$

We see that $(e_3, e_1) = 0$, $(e_3, e_2) = 0$. Continuing this process, we obtain an orthonormal set $< e_1, e_2, \ldots, e_n, \ldots >$ with the required properties.

The Conjugate Space H*

Let H be a Hilbert space and H*, its conjugate space. Let y be a fixed vector in H, Define a function f_y on H by

$$f_y(x) = (x, y) \quad \forall x \in H.$$

we assert that f_y is linear, for

$$f_y(x_1 + x_2) = (x_1 + x_2, y) \qquad \forall\, x_1, x_2 \in H$$

$$= (x_1, y) + (x_2, y)$$

$$= f_y(x_1) + f_y(x_2)$$

and $$f_y(\alpha\, x) = (\alpha\, x, y)$$

$$= \alpha(x, y) = \alpha(f_y(x))$$

Also $$|\, f_y(x)\,| = |\,(x, y)\,| \le \|\, x\,\| \,.\, \|\, y\,\|$$

(By Schwartz's Inequality)

which proves that

$$\|f_y\| \le \|\, y\,\|$$

which implies that f_y is cont. Thus f_y is linear and cont. mapping and hence is a linear functional on H. On the other hand if y = 0, then

$$f_y(x) = (x, 0) = 0 \Rightarrow \|\, f_y\,\| = \|\, y\,\|.$$

If $y \neq 0$, then

$$\|\, f_y\,\| = \sup\,\{\,|\, f_y(x)\,|\,;\, \|\, x\,\| = 1\}$$

$$\ge \left| f_y\left(\frac{y}{\|y\|}\right)\right|$$

$$\ge \left|\left(\frac{y}{\|y\|}, y\right)\right|$$

Hence

$$\| f_y \| = \| y \|$$

Thus for each $y \in H$. There is a linear functional $f_y \in H^*$ such that $\| f_y \| = \| y \|$.

Hence the mapping $y \to f_y$ is a norm preserving mapping of H into H*,

Riesz – Representation Theorem for Hilbert spaces

Theorem 7 : Let H be a Hilbert space and let f be an arbitrary functional in H*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

Proof: We shall show first that if such a y exists, then it is necessarily unique. Let y′ be another vector in H such that $f(x) = (x, y')$. Then clearly $(x, y) = (x, y')$ i.e. $(x, y - y') = 0$ for all x in H. Since zero is the only vector orthogonal to every vector, this implies that $y – y' = 0$ which implies that $y' = y$.

Now we turn to the existence of such vector y. If $f = 0$, then it clearly suffices to choose $y = 0$. We may therefore assume that $f \neq 0$. The null space $M = \{x \in H;\ f(x) = 0\}$ is thus a proper closed linear subspace of H and therefore there exists a non – zero vector y_0 in H which is orthogonal to M. We show that if α is a suitably chosen scalar, then the vector $y = \alpha\, y_0$ meets our requirements. If $x \in M$, then whatever values of α may be, we have

$$f(x) = (x, \alpha\, y_0) = 0.$$

We now choose $x = y_0$. Then we must have

$$f(y_0) = (y_0, \alpha\, y_0) = \bar{\alpha}(y_0, y_0) = \bar{\alpha}\, \| y_0 \|^2.$$

and therefore we must choose our scalar α such that

$$\bar{\alpha} = \frac{f(y_0)}{\| y \|^2} \quad \text{or} \quad \alpha = \frac{\overline{f(y_0)}}{\| y \|^2}$$

Therefore it follows that the vector $\alpha\, y_0 = \frac{\overline{f(y_0)}}{\| y \|^2} .\ y_0$ satisfies the required condition for each $x \in M$ and for $x = y_0$. Each x in H can be written in the form $x = m + \beta\, y_0$, $m \in M$. For this all that is necessary is to choose β in such a way that $f(x - \beta\, y_0) = f(x) - \beta\, f(y_0) = 0$ and this is justified by putting $\beta = \frac{f(x)}{f(y_0)}$.

Now we show that the conclusion of the theorem holds for each x in H. For this, we have

$$f(x) = f(m + \beta y_0) = f(m) + \beta f(y_0)$$
$$= (m, y) + \beta(y_0, y)$$
$$= (m + \beta y_0, y) = (x, y)$$

Remark: It follows from this theorem that the norm preserving mapping of H into H* defined by $y \to f_y$ where $f_y(x) = (x, y)$ is actually a mapping of H onto H*.

Remark: It would be pleasant if $y \to f_y$ were also a linear mapping. This is not quite true, however, for

$$f_{y_1} + f_{y_2} = f_{y_1} + f_{y_2} \text{ and } f_{\alpha y} = \bar{\alpha} f_y \qquad (1)$$

Also it follows from (1), that the mapping $y \to f_y$ is an isometry, for

$$\| f_x - f_y \| = \| f_{x-y} \| = \| x - y \|.$$

The Adjoint of an operator

Let y be a vector in a Hilbert space H and f_y its corresponding functional in H*. Operate with T* on f_y to obtain a functional $f_z = T^* f_y$ and return to its corresponding vector z in H. There are three mappings under consideration here ($H \to H^* \to H^* \to H$) and we are forming their product:

$$y \to f_y \to T^* f_y = f_z \to z \qquad (1)$$

An operator T* defined on H by

$$T^*(y) = z$$

is called adjoint of operator T.

The same symbol is used for the adjoint of T as for its conjugate because these two mappings are actually the same if H and H* are identified by means of natural correspondence. It is easy to keep track of whether T* signifies the conjugate or the adjoint of T by noticing whether it operates on functionals or on vectors.

Let x be an arbitrary vector in H. Then we have

$$(T^* f_y)(x) = f_y(T(x)) = (T(x), y)$$

and

$$(T^* f_y)(x) = f_z(x) = (x, z) = (x, T^* y)$$

so that

$$(T x, y) = (x, T^* y) \quad \text{for all x and y.}$$

The adjoint of an operator T is unique, for let T′ be another operator on H. such that

$$(T x, y) = (x, T' y) \quad \text{for all } x, y \in H.$$

$$\Rightarrow (x, T^* y) = (x, T' y)$$

$$\Rightarrow (x, T^*y - T' y) = 0.$$

$$\Rightarrow T^* y - T' y = 0 \Rightarrow T^* y = T' y \quad \forall\, y \in H.$$

$$\Rightarrow T^* = T'$$

We now prove that T* actually is an operator on H (all we know so far is that it maps H into itself) for any y and z and for all x in H, we have

$$(x, T^*(\alpha y + \beta z)) = (T x, \alpha y + \beta z)$$

$$= \bar{\alpha}(T x, y) + \bar{\beta}(T x, z)$$

$$= \bar{\alpha}(x, T^* y) + \bar{\beta}(x, T^* z)$$

$$= (x, \alpha T^* y) + (x, \beta T^* z)$$

$$= (x, \alpha T^* y + \beta T^* z)$$

Hence T* is linear. It remains to show that T* is cont. To prove this, we note that

$$\| T^* y \|^2 = (T^* y, T^* y) = (T T^* y, y)$$

$$\leq \| T T^* y \| \| y \|$$

$$\leq \| T \| \| T^* y \| \| y \|$$

which implies that $\| T^* y \| \leq \| T \| \| y \|$

for all y and therefore

$$\| T^* \| \leq \| T \|$$

Hence T* is cont. It follows therefore that $T \to T^*$ is a mapping of ß(H) into itself. This mapping is called the **adjoint operator** on ß(H).

Theorem 8 : The adjoint operator $T \to T^*$ on ß(H) has the following properties:

(1) $(T_1 + T_2)^* = T_1^* + T_2^*$

(2) $(\alpha T)^* = \bar{\alpha} T^*$

(3) $(T_1 T_2)^* = T_2^* T_1^*$

(4) $T^{**} = T$

(5) $\| T^* \| = \| T \|$

(6) $\| T^* T \| = \| T \|^2$

for all scalars α and $T_1, T, T_2 \in$ ß(H).

Proof: To prove (1), we have

$$(x, (T_1 + T_2)^* y) = ((T_1 + T_2) x, y)$$

$$= (T_1 x + T_2 x, y)$$

$$= (T_1 x, y) + (T_2 x, y)$$

$$= (x, T_1^* y) + (x, T_2^* y)$$

$$= (x, T_1^* y + T_2^* y)$$

$$= (x, (T_1^* + T_2^*) y)$$

$$\Rightarrow (T_1 + T_2)^* = T_1^* + T_2^*$$

(2) If $x \in H$, then

$$(x, (\alpha T)^* y) = (\alpha T x, y)$$

$$= \alpha(T x, y) = \alpha(x, T^* y)$$

$$= (x, \bar{\alpha} T^* y) = (x, (\bar{\alpha} T^*) y)$$

$$\Rightarrow (\alpha T)^* = \bar{\alpha} T^*$$

(3) $$(x, (T_1 T_2)^* y) = ((T_1 T_2) x, y)$$

$$= (T_1(T_2 x), y)$$

$$= (T_2 x, T_1^* y)$$

$$= (x, T_2^*(T_1^* y))$$

$$= (x, (T_2^* T_1^*) y)$$

Thus by the uniqueness of adjoint operator.

$$(T_1 T_2)^* = T_2^* T_1^*$$

(4) $$(x, T^{**} y) = (x, (T^*)^* y)$$

$$= (T^* x, y)$$

$$= \overline{(y, T^* x)} = \overline{(Ty, x)}$$

$$= (x, T y)$$

$$\Rightarrow T^{**} = T$$

(5) Let y be an arbitrary vector in H. Then

$$\| T^* y \|^2 = (T^* y, T^* y)$$

$$= (T T^* y, y)$$

$$\leq \| T T^* y \| \| y \|$$

$$\leq \| T \| \| T^* y \| \| y \|$$

$$\Rightarrow \| T^* y \| \leq \| T \| \| y \|$$

$$\Rightarrow \| T^* \| \leq \| T \|$$

Replacing T be T* in the above inequality, we have

$$\| (T^*)^* \| \leq \| T^* \|$$

$$\Rightarrow \quad \| T \| \leq \| T^* \|$$

Hence $\| T \| = \| T^* \|$

(6) To prove this equality, we have

$$\| T^* T \| \leq \| T^* \| \, \| T \| = \| T \| \, \| T \| \quad \text{[using (5)]}$$

$$= \| T \|^2$$

and

$$\| T x \|^2 = (T x, T x) = (x, T^* T x)$$

$$\leq \| x \| \, \| T^* T x \|$$

$$\leq \| x \| \, \| T^* T \| \, \| x \|$$

$$= \| x \|^2 \, \| T^* T \|$$

$$\Rightarrow \left\{ \frac{\| Tx \|^2}{\| x \|^2}, x \neq 0 \right\} \leq \| T^* T \|$$

$$\Rightarrow \sup \left\{ \frac{\| Tx \|^2}{\| x \|^2}, x \neq 0 \right\} \leq \| T^* T \|$$

$$\Rightarrow \| T \|^2 \leq \| T^* T \| \qquad (2)$$

from (1) and (2)

$$\| T^* T \| = \| T \|^2$$

Self – Adjoint Operator

Now we study some special types of operators defined on a Hilbert space. The definitions and properties of these operators depend mostly on the properties of the adjoint of an operator.

Definition: An operator A on a Hilbert space is said to be self – adjoint if it equals its adjoint i.e. if $A = A^*$.

We know that $0^* = 0$ and $I^* = I$, so zero and I are self adjoint operator. If α is real and A_1 and A_2 are self – adjoint, we claim that $A_1 + A_2$ and αA_1 are also self – adjoint. We establish these facts in the form of a more general theorem:

Theorem 9 : The self adjoint operators in ß(H) form a closed real linear subspace of ß(H) and therefore a real Banach space – which contains the identity transformation.

Proof: If A_1 and A_2 are self – adjoint and if α and β are real numbers, then

$$(\alpha A_1 + \beta A_2)^* = (\alpha A_1)^* + (\beta A_2)^*$$

$$= \bar{\alpha} A_1^* + \bar{\beta} A_2^*$$

$$= \alpha A_1 + \beta A_2.$$

[Since α, β are real and $A_1^* = A_1$, $A_2^* = A_2$.

$\Rightarrow$ $\alpha A_1 + \beta A_2$ is also self – adjoint. Therefore set of all self – adjoint operators A in ß(H) is its linear subspace.

Further, if $< A_n >$ is a sequence of self – adjoint operators which converges to an operator A, then it can be seen that A is also self – adjoint. In fact

$$\| A - A^* \| = \| A - A_n + A_n - A_n^* + A_n^* - A^* \|$$

$$\leq \| A - A_n \| + \| A_n - A_n^* \| + \| A_n^* - A^* \|$$

$$= \| A - A_n \| + \| (A_n - A)^* \|$$

$$= \| A - A_n \| + \| A_n - A \| \quad [\text{using } \| A^* \| = \| A \|]$$

$$= 2 \| A_n - A \| \to 0.$$

$$\Rightarrow A - A^* = 0 \text{ so } A = A^*.$$

Also $I^* = I$.

Hence the set of all self – adjoint operators in ß(H) form a closed linear subspace of ß(H) containing identity transformation and therefore is a real Banach space containing the identity transformation.

Theorem 10 : If A_1 and A_2 are self – adjoint operators on H, then their product $A_1 A_2$ is self – adjoint iff $A_1 A_2 = A_2 A_1$.

Proof: Suppose first that $A_1 A_2$ is self – adjoint, then

$$A_1 A_2 = (A_1 A_2)^* = A_2^* A_1^* = A_2 A_1$$

Conversely suppose that $A_1 A_2 = A_2 A_1$. Then

$$(A_1 A_2)^* = A_2^* A_1^* = A_2 A_1 = A_1 A_2$$

and therefore $A_1 A_2$ is self – adjoint.

Theorem 11 : If T is an arbitrary operator on H, then $T = 0 \Leftrightarrow (T x, y) = 0$ for all x and y.

Proof: If T = 0, then $(T x, y) = (0 x, y) = (0, y) = 0$ for all $x, y \in H$. On the other hand if $(T x, y) = 0$ for all x and y in H, then in particular $(T x, T x) = 0$ for all $x \in H$ which means that $T x = 0$ for all $x \in H$ and therefore T = 0.

Theorem 12 : If T is an operator on H, then T = 0 iff $(T x, x) = 0$ for all x.

Proof: If T = 0, then

$$(T x, x) = (0 x, x) = (0, x) = 0 \ \forall \ x \in H.$$

Conversely suppose that $(T x, x) = 0$ for all $x \in H$. We shall show that T = 0 which holds if $(T x, y) = 0$ for all $x, y \in H$. So it suffices to prove that $(T x, y) = 0$ for all $x, y \in H$. The proof of this depends on the following identity.

$$(T(\alpha x + \beta y); \alpha x + \beta y) - |\alpha|^2 (T x, x) - |\beta|^2 (T y, y)$$
$$= \alpha \bar{\beta}(T x, y) + \bar{\alpha} \beta(T y, x) \qquad (1)$$

By our hypothesis, the left side of (1) and therefore the right side as well equals zero for all α and β. If we put $\alpha = 1, \beta = 1$ in (1), we get

$$(T x, y) + (T y, x) = 0 \qquad (2)$$

and if we put $\alpha = i$ and $\beta = 1$, we get

$$i(T x, y) - i(T y, x) = 0$$

and therefore

$$(T x, y) - (T y, x) = 0 \qquad (3)$$

Adding (2) and (3), we have

$$(T x, y) = 0 \quad \text{for all } x, y \in H.$$

Hence T = 0.

Theorem 13 : An operator T on H is self adjoint iff $(T x, x)$ is real for all x.

Proof: If T is self adjoint, then

$$\overline{(Tx, x)} = (x, T x) = (x, T^* x) = (T x, x)$$

shows that (T x, x) is real for all x, On the hand, if (T x, x) is real for all x, then

$$(T\,x, x) = \overline{(Tx, x)} = \overline{(x, T^*x)} = (T^*\,x, x)$$

$$\Rightarrow ((T - T^*)\,x, x) = 0$$

$$\Rightarrow T - T^* = 0 \Rightarrow T = T^*$$

Definition: If A_1 and A_2 are self – adjoint operators on a Hilbert space H, we write $A_1 \leq A_2$ if $(A_1\,x, x) \geq (A_2\,x, x)$ for all $x \in H$.

Theorem 14 : The real Banach space of all self – adjoint operators on H is a partially ordered set whose linear structure and order structure are related by following properties :

(1) If $A_1 \leq A_2$, then $A_1 + A \leq A_2 + A$ for every A.

(2) If $A_1 \leq A_2$ then $\alpha \geq 0$, then $\alpha\,A_1 \leq \alpha\,A_2$.

Proof: Suppose B is the Banach space consisting of all self – adjoint operators on H. We define relation $\leq$ on B by

$$A_1 \leq A_2 \text{ if } (A_1\,x, x) \leq (A_2\,x, x) \qquad \forall\, x \in H,\ A_1, A_2 \in B.$$

Then

(i) $(A\,x, x) = (A\,x, x)\ \forall\ x \in H$, $A \in B$ implies $A \leq A\ \forall\ A \in B$. Hence $\leq$ is reflexive.

(ii) If $A_1, A_2 \in B$ such that $A_1 \leq A_2$ and $A_2 \leq A_1$, then

$$A_1 \leq A_2 \Rightarrow (A_1\,x, x) \leq (A_2\,x, x)$$

$$A_2 \leq A_1 \Rightarrow (A_2\,x, x) \leq (A_1\,x, x)$$

Combining these two expressions, we have

$$(A_1\,x, x) = (A_2\,x, x)$$

$$\Rightarrow ((A_1 - A_2)x, x) = 0 \Rightarrow A_1 - A_2 = 0$$

$$\Rightarrow A_1 = A_2.$$

Therefore the relation $\leq$ is anti – symmetric.

(iii) Let $A_1, A_2, A_3 \in B$ such that $A_1 \leq A_2$ and $A_2 \leq A_3$. Then

$$A_1 \leq A_2 \Rightarrow (A_1\,x, x) \leq (A_2\,x, x)$$

$$A_2 \leq A_3 \Rightarrow (A_2\,x, x) \leq (A_3\,x, x)$$

On both of these yield

$$(A_1 x, x) \leq (A_3 x, x)$$

$$\Rightarrow A_1 \leq A_3.$$

Thus the relation is transitive.
Hence ≤ is a partially ordered relation. Now we prove the relation (1) and (2)

(1) $A_1 \leq A_2 \Rightarrow (A_1 x, x) \leq (A_2 x, x)$

$$\Rightarrow (A_1 x, x) + (A x, x) \leq (A_2 x, x) + (A x, x)$$

$$\Rightarrow ((A_1 + A) x, x) \leq ((A_2 + A) x, x)$$

$$\Rightarrow A_1 + A \leq A_2 + A$$

(2) $A_1 \leq A_2 \Rightarrow (A_1 x, x) \leq (A_2 x, x)$

$$\Rightarrow \alpha (A_1 x, x) \leq \alpha (A_2 x, x)$$

$$\Rightarrow (\alpha A_1 x, x) \leq (\alpha A_2 x, x)$$

$$\Rightarrow ((\alpha A_1) x, x) \leq ((\alpha A_2) x, x)$$

$$\Rightarrow \alpha A_1 \leq \alpha A_2 \quad \forall \alpha \geq 0.$$

Hence theorem.

Positive Operator

Definition: A self – adjoint operator A is said to be **positive** if $A \geq 0$, i.e. $(A x, x) \geq 0$ for all x.

It is clear that 0 and I are positive, as are T* T and T T* for an arbitrary operator T.

Theorem 15 : If A is a positive operator on H, then I + A is non – singular. In particular I + T* T and I + T T* are non – singular for an arbitrary operator T on H.

Proof: We must show that I + A is one to one onto as a mapping of H into itself. First of all we observe that

$$(I + A)(x) \Rightarrow x + A x = 0$$

$$\Rightarrow A x = - x \Rightarrow (A x, x) = (-x, x) \geq 0.$$

$$\Rightarrow - \| x \|^2 \geq 0 \Rightarrow x = 0 \ \forall \ x \in H.$$

Then

$$(I + A)(x) = (I + A)y \Rightarrow (I + A)(x - y) = 0.$$

$$\Rightarrow x - y = 0 \Rightarrow x = y$$

$\Rightarrow$ I + A is one – to – one.

It remains to show that I + A is onto. It is sufficient to prove that range of I + A equals H. Let M be the range of I + A.
Then

$$\| (I + A)\, x \|^2 = \| x + A\, x \|^2 = (x + A\, x, x + A\, x)$$

$$= (x, x) + (x, A\, x) + (A\, x, x) + (A\, x, A\, x)$$

$$= \| x \|^2 + 2(A\, x, x) + \| A\, x \|^2$$

[Since (A x, x) is real]

$$\geq \| x \|^2$$

$$\Rightarrow \| x \|^2 \leq \| (I + A)\, x \|^2.$$

By this inequality and the completeness of H, M is complete and therefore closed. Suppose that $M \subset H$. Then $\exists$ a non – zero vector $x_0 \perp M$ such that

$$(x_0, (I + A)\, x_0) = 0$$

$$\Rightarrow (x_0, x_0) + (x_0, A\, x_0) = 0$$

$$\Rightarrow \| x_0 \|^2 + (A\, x_0, x_0) = 0$$

$$\Rightarrow \| x_0 \|^2 = -(A\, x_0, x_0) \leq 0$$

$$\Rightarrow x_0 = 0.$$

which contradicts the fact that x_0 is a non – zero vector.
Hence M = H. It follows therefore that I + A is one – to – one and onto and hence non – singular.

Normal Operator

Definition: An operator N on a Hilbert space H is said to be **normal** if it commutes with its adjoint that is N N* = N* N.

Theorem 16 : The set of all normal operators on H is a closed subset of ß(H) which contains the set of all self – adjoint operator and is closed under scalar multiplication.

Proof: If N is a self – adjoint operator, then

$$N^* = N \Rightarrow N N^* = N^* N.$$

Thus it follows that every self – adjoint operator is normal. Therefore the set M contains the set of all self – adjoint operators.

Let α be a scalar and N a normal operator, then

$$(\alpha N)(\alpha N)^* = (\alpha N)(\bar{\alpha} N^*) = \alpha \bar{\alpha}(N N^*)$$

$$= \bar{\alpha} \alpha (N^* N)$$

$$= (\bar{\alpha} N^*)(\alpha N)$$

$$= (\alpha N)^* (\alpha N)$$

Now consider the set M of all normal operators on H. It is clearly a subset of ß(H). To show that it is closed, it is sufficient to prove that every Cauchy sequence $\{N_k\}$ of normal operators on H converges to a normal operator. Due to the completeness of ß(H) this sequence converges to some operator N we shall show that N is normal. Since $N_k^* \to N^*$, we have

$$\| N N^* - N^* N \| = \| N N^* - N_k N_k^* + N_k N_k^* - N_k^* N_k + N_k^* N_k - N^* N \|$$

$$\leq \|NN^* - N_kN_k^*\| + \|N_kN_k^* - N_k^*N_k\| + \|N_k^*N_k - N^*N\|$$

$$= \|NN^* - N_kN^*\| + \|N^*_kN_k - N^*\| \to 0$$

$$\leq \| N N^* - N_k N^* \| + \| N_k^* N_k \| - N^* N \| \to 0$$

which implies that

$$N N^* - N^* N = 0$$

$$\Rightarrow N N^* = N^* N$$

therefore N is normal.

Theorem 17 : If N_1 and N_2 are normal operators on a Hilbert space H with the property that either commutes with the adjoint of the other, then $N_1 + N_2$ and $N_1 N_2$ are normal.

Proof: We are given that

$$N_1 N_1^* = N_1^* N_1,\ N_2 N_2^* = N_2^* N_2$$

$$N_1 N_2^* = N_2^* N_1,\ N_2 N_1^* = N_1^* N_2$$

We show first that $N_1 + N_2$ is normal. For this, we have.

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1 + N_2)(N_1^* + N_2^*)$$

$$= N_1 N_1^* + N_1 N_2^* + N_2 N_2^* + N_2 N_1^*$$

$$= N_1^* N_1 + N_2^* N_1 + N_1^* N_2 + N_2^* N_2$$

$$= (N_1^* + N_2^*)(N_1 + N_2)$$

$$= (N_1 + N_2)^* (N_1 + N_2)$$

which shows that $N_1 + N_2$ is normal.
Similarly

$$(N_1\ N_2)(N_1\ N_2)^* = (N_1\ N_2)(N_2^*\ N^*_1)$$

$$= N_1 (N_2 N_2^*) N_1^*$$

$$= N_1 (N_2^* N_2) N_1^*$$

$$= (N_1 N_2^*)(N_2 N_1^*)$$

$$= (N_2^* N_1)(N_1^* N_2)$$

$$= N_2^* (N_1 N_1^*) N_2$$

$$= (N_2^*(N_1^* N_1) N_2$$

$$= (N_2^* N_1^*)(N_1 N_2)$$

$$= (N_1 N_2)^* (N_1 N_2)$$

$\Rightarrow N_1 N_2$ is normal.

Theorem 18 : An operator on a Hilbert space H is **normal** if and only if

$$\| T^* x \| = \| T x \| \text{ for every } x.$$

Proof: T is normal iff

$$T T^* = T^* T \Leftrightarrow T T^* - T^* T = 0$$

$\Rightarrow ((T\,T^* - T^*\,T)\,x, x) = 0 \quad \forall\, x \in H$

[since an operator T on H is zero iff (T x, x) = 0]

$\Leftrightarrow (T\,T^*\,x, x) = (T^*\,T\,x, x)$

$\Leftrightarrow (T^*\,x, T^*\,x) = (T\,x, T\,x)$

$\Leftrightarrow \| T^*\,x \|^2 = \| T\,x \|^2$

$\Leftrightarrow \| T^*\,x \| = \| T\,x \|$

Theorem 19 : If N is a normal operator on H, then

$\| N^2 \| = \| N \|^2$ [by the above theorem]

Proof: Since N is normal, we have

$$\| N^*\,x \| = \| N\,x \| \qquad \forall\, x \in H \qquad (*)$$

$$\Rightarrow \| N^2 \| = \sup \{ \| N^2\,x \|; \| x \| \le 1 \}$$

$$= \sup \{ \| N(N\,x) \|; \| x \| \le 1 \}$$

$$= \sup \{ \| N^*(N\,x) \|; \| x \| \le 1 \}$$

$$= \sup \{ \| N^*\,N\,x \|; \| x \| \le 1 \}$$

[By the property of adjoint operation on ß(H)]

Remark: For an arbitrary operator T on a Hilbert space, we form

$$A_1 = \frac{T + T^*}{2}, \qquad A_2 = \frac{T - T^*}{2i}$$

It can be shown that A_1 and A_2 are self adjoint and they have the property that $T = A_1 + i\,A_2$

In fact $\qquad A_1^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + T)$

$$= \frac{T + T^*}{2} = A_1 \Rightarrow A_1 \text{ is self – adjoint}$$

and $A_2^* = [\frac{1}{2i}(T - T^*)]^* = -\frac{1}{2i}(T^* - T)$

$$= \frac{1}{2i}(T - T^*) = A_2$$

$\Rightarrow A_2$ is self – adjoint.

$$A_1 + i\,A_2 = \frac{T + T^*}{2} + \frac{T - T^*}{2} = T$$

Also $\quad T^* = (A_1 + i\,A_2)^* = A_1^* - i\,A_2^*$

$$= A_1^* - i\,A_2^* = A_1 - i\,A_2\,.$$

A_1 and A_2 are called real and imaginary parts of T.

Theorem 20 : If T is an operator on H, then T is normal $\Leftrightarrow$ its real and imaginary parts commute.

Proof: If A_1 and A_2 are real and imaginary parts of T so that $T = A_1 + i\,A_2$ and

$$T^* = A_1 - i\,A_2, \text{ then}$$

$$T\,T^* = (A_1 + i\,A_2)\,(A_1 - i\,A_2) = A_1^2 + A_2^2 + i(A_2\,A_1 - A_1\,A_2)$$

and $\quad T^*\,T = (A_1 - i\,A_2)\,(A_1 + i\,A_2) = A_1^2 + A_2^2 + i\,(A_1\,A_2 - A_2\,A_1)$

It is clear that if $A_1\,A_2 = A_2\,A_1$

Then $\quad T\,T^* = T^*\,T$

Conversely T is normal iff $T\,T^* = T^*\,T$

$$\Leftrightarrow A_1\,A_2 - A_2\,A_1 = A_2\,A_1 - A_1\,A_2$$

$$\Leftrightarrow 2\,A_1\,A_2 = 2\,A_2\,A_1$$

$$\Leftrightarrow A_1\,A_2 = A_2\,A_1\,.$$

Unitary Operator

Definition: An operator $\cup$ on H is said to be **unitary** if $\cup\,\cup^* = \cup^*\,\cup = I$

Theorem 21 : If T is an operator on H, then the following conditions are all equivalent to one another.

(1) $T^*\,T = I$

(2) $(T\,x, T\,y) = (x, y)$ for all x and y

(3) $\|\,T(x)\,\| = \|\,x\,\|$ for all x.

Proof: (1) $\Rightarrow$ (2),

If T* T = I, then

$$(T\,x, T\,y) = (x, T^*\,T\,y) = (x, I\,y) = (x, y)$$

for all x and y

(2) $\Rightarrow$ (3). If (T x, T y) = (x, y) for all x and y, then taking y = x, we have

$$(T\,x, T\,x) = (x, x) = \|x\|^2$$

$$\Rightarrow \|T(x)\|^2 = \|x\|^2$$

$$\Rightarrow \|T(x)\| = \|x\| \qquad \forall\, x.$$

(3) $\Rightarrow$ (1) when $\|T(x)\| = \|x\|$

$$\Rightarrow \|T(x)\|^2 = \|x\|^2$$

$$\Rightarrow (T\,x, T\,x) = (x, x)$$

$$\Rightarrow (T^*\,T\,x, x) = (I\,x, x)$$

$$\Rightarrow ((T^*\,T - I)\,x, x) = 0 \qquad \forall\, x \in M$$

$$\Rightarrow T^*\,T - I = 0 \Rightarrow T^*\,T = I.$$

Theorem 22 : An operator T on H is **unitary** iff it is an isometric isomorphism of H onto itself.

Proof: If T is unitary, then we know from the definition that it is onto.

Moreover since T* T = I, by the previous Theorem.

$$\|T(x)\| = \|x\| \qquad \forall\, x \in H.$$

Hence T is an isometric isomorphism of H onto itself.

Conversely if T is an isometric isomorphism of H onto itself, then T is a one – one mapping onto H such that

$\|T(x)\| = \|x\|\ \forall\, x \in H$ and so by the above theorem, T* T = I

Since T is an isometric isomorphism of H onto itself, T^{-1} exists and then

$$T^*\,T = I \Rightarrow T^* = T^{-1}.$$ Also we note that

$$T\,T^* = T\,T^{-1} = I$$

$$\Rightarrow T^*\,T = T\,T^* = I \Rightarrow T \text{ is unitary.}$$

Projections

We know that a projection on a Banach space B is an idempotent operator on B i.e. an operator P with the property $P^2 = P$. It was proved that each projection P. determines a pair of closed linear subspaces M and N, the range and null space of P such that $B = M \oplus N$ and also conversely that each such pair of closed linear subspaces M and N determines a projection P with range M and null space N.

The structure which a Hilbert space H enjoy in addition to being a Banach space enables to single out for special attentions those projections whose range and null space are orthogonal.

We establish the following theorem:

Theorem 23 : If P is a projection on H with range M and null space N, then $M \perp N$, $\Leftrightarrow$ P is self – adjoint and in this case $N = M^{\perp}$.

Proof: Since P is projection on a Hilbert space H with range M and null space N, we have $H = M \oplus N$, so each vector $z \in H$ can be written uniquely in the form $z = x + y$, $x \in M$, $y \in N$.

If $M \perp N$, then $(x, y) = (y, x) = 0$. Therefore for all z in H, we have

$$(P^* z, z) = (z, P z) = (z, x) = (x + y, x)$$

$$= (x, x) + (y, x) = (x, x).$$

and

$$(P z, z) = (x, z) = (x, x + y) = (x, x) + (x, y)$$

$$= (x, x)$$

$$\Rightarrow (P^* z, z) = (P z, z)$$

$$\Rightarrow [(P^* - P) z, z] = 0$$

$$\Rightarrow P^* - P = 0 \Rightarrow P^* = P .$$

Conversely suppose that $P^* = P$, to prove that $M \perp N$, it is sufficient to show that if x and y are arbitrary elements of M and N respectively, then $(x, y) = 0$.

In fact we have,

$$(x, y) = (P\,x, y) = (x, P^*\,y) = (x, P\,y)$$
$$= (x, 0) = 0. \qquad \{\text{ N is the null space } y \in N, P(y) = 0\}$$

Hence $M \perp N$.

It remains to prove that if $M \perp N$. Then $N = M^{\perp}$. It is clear that $N \subseteq M^{\perp}$ and if N is a proper subset of $M^{\perp}$ and therefore a proper closed linear subspace of the Hilbert space $M^{\perp}$, there exists a non – zero vector z_0 in $M^{\perp}$ such that $z_0 \perp N$. Since $z_0 \perp M$ and $z_0 \perp N$ and $H = M \oplus N$. It follows that $z_0 \perp H$. This is impossible and hence $N = M^{\perp}$.

Definition: A projection on H whose range and null space are orthogonal is called a prependicular projection.

The only projections considered in the theory of Hilbert spaces are those which are perpendicular.

In the light of above theory an operator P on a Hilbert space H is a perpendicular projection if $P^2 = P$ and $P^* = P$.

Moreover P is projection on M only if $(I - P)$ is a projection on $M^{\perp}$.

Theorem 24 : If P and Q are the projections on closed linear subspaces M and N of H .
Then $M \perp N \Leftrightarrow P\,Q = 0 \Leftrightarrow Q\,P = 0$.

Proof: If $M \perp N$, then $N \subseteq M^{\perp}$. Since Q is a projection on N, Qz is in N for each $z \in H$.

Therefore $Qz \in M^{\perp} \Rightarrow P(Q\,z) = 0$

$$\Rightarrow P\,Q(z) = 0 \Rightarrow P\,Q = 0.$$

Moreover taking adjoint, we have

$$\mathbf{P\,Q = 0 \Rightarrow (P\,Q)^* = 0^*}$$

$$\Rightarrow Q^*\,P^* = 0 \Rightarrow Q\,P = 0.$$

Hence $M \perp N \Rightarrow P\,Q = 0 \Rightarrow Q\,P = 0$.

Conversely suppose that Q P = 0

$\Rightarrow P\,Q = 0$, then for $x \in M$ or $y \in N$, we have

$$(x, y) = (P\,x, Q\,y) = (x, P^*\,Q\,y)$$

$$= (x, P\,Q\,y) = (x, 0.y) = (x, 0) = 0.$$

Hence $M \perp N$.
Therefore $Q\,P = 0 \Rightarrow P\,Q = 0 \Rightarrow M \perp N$.

Definition: Two projections P and Q are orthogonal if P Q = 0.

Theorem 25 : If $P_1, P_2, \ldots\ldots\ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots\ldots\ldots, M_n$ of H, then $P = P_1 + P_2 + \ldots\ldots\ldots + P_n$ is a projection $\Leftrightarrow$ P_i 's are pairwise orthogonal (in the sense that $P_i\,P_j = 0$ whenever $i \neq j$) and in this case, P is the projection on $M = M_1 + M_2 + \ldots\ldots\ldots + M_n$.

Proof: Each P_i is a perpendicular projection therefore $P_i^* = P_i = P_i^2$ for i = 1, 2, ……, n.
Then

$$P^* = (P_1 + P_2 + \ldots\ldots\ldots\ldots + P_n)^* = P_1^* + P_2^* + \ldots\ldots\ldots + P_n^*$$

$$= P_1 + P_2 + \ldots\ldots\ldots\ldots + P_n = P.$$

Hence P is self – adjoint. Now P is a projection $\Leftrightarrow$ it is idempotent.

If P_i's are pairwise orthogonal, then

$$P_i\,P_j = 0 \qquad \text{for } i \neq j$$

Hence

$$P^2 = (P_1 + P_2 + \ldots\ldots\ldots + P_n)^2$$

$$= \sum_{i=1}^{n} P_i^2 + 2 \sum_{i \neq 1} P_i\,P_j$$

$$= \sum_{i=1}^{n} P_i \qquad [\because P_i^2 = P_i \text{ and } P_i\,P_j = 0]$$

$$= P$$

$\Rightarrow$ P is idempotent.

Thus we have proved that if P_i's are pairwise orthogonal, then P is a projection.

To prove the converse we assume that P is idempotent. Let x be a vector in the range of P_i so that $P_i(x) = x$.
Then

$$\|x\|^2 = \|P_i(x)\|^2 \leq \sum_{j=1}^{n} \|P_j(x)\|^2 = \sum_{j=1}^{n} (P_j\,x, P_j(x)) = \sum_{j=1}^{n} (P_j\,x, P_j^*\,x)$$

$$= \sum_{j=1}^{n} (P_j^2 x, x)$$

$$= \sum_{j=1}^{n} (P_j x, x)$$

$$= [(P_1 + P_2 + \ldots\ldots + P_n) x , x]$$

$$= (P x, x) = (P^2 x, x)$$

$$= (P x, P^* x)$$

$$= (P x, P x) = \| P x \|^2 \leq \| x \|^2$$

Since $\| x \|^2 = \| P x + (I - P) x \|^2$

$= \| P x \|^2 + \| (I - P) x \|^2$ [Pythagorean theorem]

$\Rightarrow \| P(x) \|^2 \leq \| x \|^2$

Hence $\| x \|^2 \leq \sum_{j=1}^{n} \| P_j(x) \|^2 \leq \| x \|^2 \Rightarrow \| x \|^2 = \sum_{j=1}^{n} \| P_j(x) \|^2$

$\sum_{j=1}^{n} \| P_j x \|^2 = \| P_i (x) \|^2 = \| x \|^2$ [Since $\| P_i(x) \|^2 = \| x \|^2$]

which implies that $\| P_j(x) \| = 0$ for $j \neq i$.

Now $P_j(x) = 0 \Rightarrow x \in$ Null space of P_j for $j \neq i$. Thus range of P_i is contained in the null space of P_j i.e. $M_i \subseteq M_j^{\perp}$ for every $i \neq j$ and this means that $M_i \perp M_j$ for $i \neq j$. Hence [by the preceding theorem] P_i's are pairwise orthogonal.

We now show that P is a projection on M. Firstly we observe that since $\| P(x) \| = \| x \| \ \forall \ x \in M_i$, each M_i is contained in the range of P and therefore $M = \sum_{i=1}^{m} M_i$ is also contained in the range of P.

Secondly if x is a vector in the range of P , then

$$x = P x = (P_1 + P_2 + \ldots\ldots + P_n) x.$$

$$= P_1 x + P_2 x + \ldots\ldots + P_n x.$$

is evidently in $M = \sum_{i=1}^{m} M_i$ since $P_i x \in M_i$.

Hence the theorem.

Definition: A closed linear subspace M of a Hilbert space H is said to be invariant under an operator T on H if $T(M) \subseteq M$.

If both M and $M^{\perp}$ are invariant under T, then we say that M reduces T (or that T is reduced by M)

Theorem 26 : A closed linear subspace M of H is invariant under an operator $T \Leftrightarrow M^{\perp}$ is invariant under T*.

Proof: Suppose first that M is invariant under an operator T, then $T(x) \in M$ for all $x \in M$. We shall show that $M^{\perp}$ is invariant under T*. If y is any vector of $M^{\perp}$,

Then $(x, y) = 0$ for all $x \in M$.

$(x, T^* y) = (T x, y) = 0$ since $T x \in M$.

$\Rightarrow T^* y \in M^{\perp}$ for all y $y \in M^{\perp}$

Hence $M^{\perp}$ is invariant under T*.

Conversely suppose that $M^{\perp}$ is invariant under T*. Then $M^{\perp\perp}$ is invariant under $(T^*)^* = T^{**}$. But $M^{\perp\perp} = M$ and $T^{**} = T$,

Therefore it follows that M is invariant under T.

Theorem 27 : A closed linear subspace M of H reduces an operator $T \Leftrightarrow$ M is invariant under both T and T*.

Proof: By definition we know that M reduces T
$\Leftrightarrow$ M is invariant under T and $M^{\perp}$ is invariant under T

$\Leftrightarrow$ M is invariant under T and $M^{\perp\perp}$ is invariant under T* [By previous Theorem].

$\Leftrightarrow$ M is invariant under both T and T*.

Theorem 28 : If P is a projection on a closed linear subspace M of H, then M is invariant under an operator $T \Leftrightarrow T P = P T P$

Proof: If M is invariant under T and x is an arbitrary vector in H, then

$$x \in H \Rightarrow P(x) \in M \Rightarrow T(P(x)) \subset M$$

$$\Rightarrow T\,P\,(x) \in M$$

$$\Rightarrow P(T\,P\,(x)) = T\,P(x)$$

$$\Rightarrow (P\,T\,P)\,(x) = T\,P\,(x)$$

$$\Rightarrow P\,T\,P = T\,P.$$

conversely if T P = P T P and x is a vector in M then P(x) = x

$$\Rightarrow T(P(x)) = T(x)$$

$$\Rightarrow P\,T(P(x)) = T(x)$$

But P T P (x) $\in$ M, therefore T(x) $\in$ M.

Hence M is invariant under T.

Theorem 29 : If P is the projection on a closed linear subspaces M of H, then M reduces an operator T $\Leftrightarrow$ T P = P T.

Proof: By a result proved above, M reduces T iff M is invariant under T and

T* iff T P = P T P and T* P = P T* P

$\Leftrightarrow$ T P = P T P and (T* P)* = (P T* P)*

$\Leftrightarrow$ T P = P T P and

$$P^*\,T^{**} = P^*\,T^{**}\,P^* \Leftrightarrow T\,P = P\,T\,P$$

And P T = P T P [$\because$ P* = P and T** = T]

$$\Leftrightarrow T\,P = P\,T.$$

Reflexivity of Hilbert space

Let H be a Hilbert space with inner product denoted by (y, x). The dual (conjugate space) H* is then a Hilbert space with inner product given by (x*, y*) = (y, x) for each x* and y* in H* where x $\to$ x* and y $\to$ y* under the mapping H $\to$ H*.

We now establish the following result concerning the reflexivity of a Hilbert space.

Theorem 30 : Every Hilbert space is reflexive.

Proof: Let H* denote the dual space of a Hilbert space H. Consider the mapping T defined by

$$T : H \to H^*$$

$$y \to T\,y = f \qquad (1)$$

where the bounded linear functional f is, for any $x \in X$, given by

$$(T\,y)\,(x) = f(x) = (x, y) \qquad (2)$$

Suppose now that under T,

$$y_1 \to f_1$$

and $$y_2 \to f_2$$

and let $y_1 + y_2 \to g$.
Thus

$$g(x) = (x, y_1 + y_2)$$

$$= (x, y_1) + (x, y_2)$$

$$= f_1(x) + f_2(x)$$

and we conclude that

$$T(y_1 + y_2) = T(y_1) + T(y_2)$$

Showing that T is additive. Now suppose under T, $y \to f$
And for a scalar α, let $T(\alpha\, y) = h$, then

$$h(x) = (x, \alpha\, y) = \bar{\alpha}(x, y) = \bar{\alpha}\, f(x),$$

therefore

$$T(\alpha\, y) = \bar{\alpha}\, T(y)$$

Showing that T is conjugate linear. Also, by Riesz – Representation theorem for bounded linear functionals on a Hilbert space, to each bounded linear function f, there exists a unique $y \in H$ such that for every $x \in H$, $f(x) = (x, y)$ and $\| f \| = \| y \|$. In view of this the mapping T is onto and further

$$\| f \| = \| T\,y \| = \| y \| \; (y \to T\,y = f)$$

Therefore T is norm – preserving mapping or isometry. As we know that an isometry is always a 1 – 1 mapping.

Thus we have, the mapping T constitutes a 1 – 1 onto isometric, conjugate linear mapping from a Hilbert space onto conjugate space. Thus we see that Hilbert space and their conjugate spaces are indistinguishable metrically and almost indistinguishable algebraically. [Almost because T is conjugate linear]

Let x* be a bounded linear functional on H and $x \in H$. Denote $x^*(x) = [x, x^*]$. Consider the mapping

$$J : H \to H^{**}$$

$$x \to x^{**}$$

where for defining equation for Jx we have for any $x^* \in H^*$

$$x^{**}(x^*) = [x^*, x^{**}] = [x^*, x] = [x, x^*] = x^*(x) \qquad (3)$$

we now show that x** is a bounded linear functional. Let $x^* \in X^*$, then

$$|\, x^{**}(x^*) \,| = |\, x^*(x) \,| \le |\, x^* \,\|\, \|\, x \,\|$$

$$\Rightarrow \|\, x^{**} \,\| \le \|\, x \,\| \qquad (*)$$

Further if x = 0, then

$$0 \le \|\, x^{**} \,\| \le 0 .$$

And consequently $\|\, x^{**} \,\| = \|\, x \,\| = 0$

If x is a non zero vector, then there must be some bounded linear functional x_0^* with norm 1 s. That $x_0^*(x) = \|\, x \,\|$. But

$$\|\, x^{**} \,\| = \sup_{\|x^*\|=1} |\, x^{**}(x^*) \,| = \sup_{\|x^*\|=1} |\, x^*(x) \,|$$

$$\ge |\, x^*(x) \,| = \|\, x \,\| \qquad (**)$$

Thus $\|\, x^{**} \| = \|x\|$

$\Rightarrow$ J is an isometry. Since isometry is always a 1-1 mapping, it follows that J is an isomorphism. It remains to show that J is onto. To this end, let f be an element of H**. We must find $z \in H$ such that Jz = f. For T defined in (1) consider the functional g defined by

$$g : H \to f$$

$$x \to \overline{f(T(x))}$$

For $x_1, x_2 \in H$, consider

$$g(x_1 + x_2) = \overline{f(T(x_1 + x_2)}$$

$$= \overline{f(Tx_1 + Tx_2)}$$

$$= \overline{f(T(x_1))} + \overline{f(T(x_2))}$$

$$= g(x_1) + g(x_2) \qquad (4)$$

$\Rightarrow$ g is additive.

Now let $x \in H$, $\alpha \in F$, then

$$g(\alpha\, x) = \overline{f(T\alpha\, x)} = \overline{f(\bar{\alpha}T(x))}$$

$$= \overline{\bar{\alpha}\, f(T(x)} = \alpha.\, g(x)$$

Hence g is linear.
Further since T is an isometry, we have

$$|\, g(x)\, | = |\, \overline{f(T(x))}\, | = |\, f(T\, x)\, | \le \|\, f\, \|\, \|\, T\, x\, \|$$

$$= \|\, f\, \|\, \|\, x\, \|$$

Thus g is bounded.
By Riesz – Representation Theorem, $\exists$ $z \in H$ such that for all $x \in H$, $g(x) = (x, z)$

Or $\overline{f(Tx)} = (x, z)$

$$\Rightarrow f(T\, x) = (z, x) \qquad (5)$$

On the other hand by the definition of J and T (using(2) and (3)

$$(J\, z)\, (T\, x) = z^{**}\, (T\, x) = T\, x(z) = (z, x) \qquad (6)$$

Thus (5) and (6) yield that Jz and f agree on every member of H*. Hence they are same. This completes the proof.

Example: Show that a Hilbert space is finite dimensional $\Leftrightarrow$ every complete orthonormal set is a basis.

Solution: Let H be a finite dimensional Hilbert space of dimensional n. Let S = < e_i > be a complete orthonormal set in H. Then we have to show that S is a basis for H. Since S is an orthonormal set, therefore it is linearly independent.

Also S must be a finite set because it can not contain more than n vectors. [Since H is finite dimensional]. Now let x $\in$ H. Since S is a complete orthonormal set, therefore we have x = $\sum_{e_i \in S}$ (x, e_i) e_i. Thus each vector x in H can be written as linear combination of vectors in the set S and so S generates H. Therefore S is a basis for H. [Thus in a finite dimensional Hilbert space of dimension n every complete orthonormal set must contain exactly n vectors].

Conversely suppose that every complete orthonormal set in a Hilbert space H is a basis for H. Then to show that H is finite dimensional. Let S be a complete orthonormal set in H. Then by hypothesis S is a basis for H. We are to show that S is infinite set. Suppose δ is infinite. Then we can certainly extract a denumerable sequence of distinct points of S

$$e_1, e_2, e_3, \ldots\ldots\ldots, e_n, \ldots\ldots\ldots..$$

Consider now the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e_n.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent, $\Rightarrow$ the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ e_n is convergent [by the result that. Let H be a Hilbert space and let S = < e_1, e_2,……,e_n,…..> be countably infinite orthonormal set in H. Then a series of the form $\sum_{n=1}^{\infty}$ $\alpha_n e_n$ is convergent iff $\sum_{n=1}^{\infty}$ $|\alpha_n|^2 < \infty$.

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ e_n must converge to some vector x in H. Since S is a basis for H, therefore we can write x as some finite linear combination of vectors in S. Let

$$x = \alpha_\lambda e_\lambda + \ldots\ldots + \alpha_\mu e_\mu$$

where e_λ,….. $e_\mu \in$ S and α_λ, ………., α_μ are scalars. Let j be any +ve integer having value different from the values of indices λ,………, μ. We have

$$(x, e_j) = (\alpha_\lambda e_\lambda + \ldots\ldots + \alpha_\mu e_\mu, e_j)$$

$$= \alpha_\lambda(e_\lambda, e_j) + \dots\dots\dots \alpha_\mu(e_\mu, e_j) = 0.$$

Also $(x, e_j) = (\sum_{n=1}^{\infty} \frac{1}{n^2}\ e_n, e_j)$ [$\because$ $x = \sum_{n=1}^{\infty} \frac{1}{n^2}\ e_n$]

$$= \frac{1}{n^2}$$

Thus we have $\frac{1}{n^2} = 0$ which is not possible. Therefore the set S must be finite and H is finite dimensional.

Theorem 31 : Prove that any two complete orthonormal sets in a Hilbert space H have the same cardinal number.

Proof: Let S_1 and S_2 be two complete orthonormal sets in a Hilbert space H. Suppose one of these sets is finite. Let S_1 be finite and $S_1 = \{e_1, e_2, \dots\dots\dots, e_n\}$. Since S_1 is an orthonormal set, therefore it is linearly independent. Also since S_1 is complete, therefore if $x \in H$, then we have

$$x = \sum_{i=1}^{n} (x, e_i)\ e_i$$

Thus S_1 generates H. Therefore S_1 is a basis for H and so H is finite dimensional and dim H = n. Since S_2 is also a complete orthonormal set in H, therefore S_2 must also be a basis for H. Since S_1 and S_2 are both bases for H, therefore they must have the same number of elements.

Now let us suppose that both S_1 and S_2 are infinite sets. Let $x \in S_1$ and let $S_2(x) = \{y: y \in S_2$ and $(y, x) \neq 0\}$. Then $S_2(x)$ is a subset of S_2 and thus $S_2(x)$ is a countable set \. Let z be any arbitrary member of S_2. Since S_1 is a complete orthonormal set and therefore by Parseval's identity, we have

$$\| z \|^2 = \sum_{x \in S_1} | (z, x) |^2$$

But $z \in S_2 \Rightarrow z$ is a unit vector.
Therefore we have

$$1 = \sum_{x \in S_1} | (z, x) |^2.$$

From this relation we see that there must exist some vector $x \in S_1$ such that $(z, x) \neq 0$. Then by our definition of $S_2(x)$, we have $z \in S_2(x)$. Thus $z \in S_2 \Rightarrow z \in S_2(x)$ for some $x \in S_1$. Therefore we have

$$S_2 = \bigcup_{x \in S_1} S_2(x) \qquad (1)$$

Let n_1, n_2 be the cardinal numbers of S_1, S_2 respectively. Since the cardinal number of the union of an arbitrary collection of sets can not exceed the cardinal number of index set, therefore $n_2 \le n_1$. Interchanging the roles of S_1 and S_2 we get $n_1 \le n_2$.
Therefore we have $n_1 = n_2$

Remark: Let S be a complete orthonormal set in a Hilbert space H. Then the cardinal number of S is said to be the orthogonal dimension of H. If H has no complete orthonormal set i.e. if H is the zero space, then the orthogonal dimensional of H is said to be zero.

Definition: Operators S and T are said to be metrically equivalent if $\| S x \| = \| T x \| \ \forall \ x \in H$.

Theorem 32 : Operators S and T are metrically equivalent if $S^* S = T^* T$

Proof: Let S and T be metrically equivalent

$$\| S x \| = \| T x \| \qquad \forall \ x \in H.$$

$$\Leftrightarrow (S^*. Sx, x) = (S x, S x) = \| S x \|^2 = \| T x \|^2$$

$$= (T x, T x) = (T^* T x, x)$$

$$\Rightarrow ((S^* S - T^* T) x , x) = 0$$

$$\Rightarrow S^* S - T^* T = 0$$

$$\Rightarrow S^* S = T^* T.$$

Theorem 33 : An operator T is normal iff T and T* are metrically equivalent.

Proof: Suppose T is normal $\Rightarrow T^* T = T T^*$
and so by the above theorem, T* and T are metrically equivalent.
Conversely suppose that T and T* are metrically equivalent

$$\Rightarrow \| T^* x \| = \| T x \|$$

$$\Rightarrow T^* T = T T^*$$

$$\Rightarrow T \text{ is normal.}$$

Finite Dimensional Spectral Theory

First we give basic definitions and results.

Definition 1: Let T be an operator on a Hilbert space H. A vector $x \in H$ is said to be a proper vector (eigen-vector , latent vector or characteristic vector) for the operator T if (i) $x \neq 0$ and (ii) $Tx = ux$ for a suitable scalar u. if also $Tx = vx$, then $Tx = ux$ and $Tx = vx$ implies $(u - v)x = 0$. Since $x \neq 0$, it follows that $u = v$. Thus a proper vector x determines uniquely the associated scalar u.

Definition 2: A scalar u is said to be a proper value (Eigen value , latent root or characteristic root(value)) for the operator T in case there exists a non-zero vector x such that $Tx = ux$.

Thus u is a proper value for T if and only if the null space of T-uI is not equal to [0].

Remark : If the Hilbert space H has no non-zero vector at all , then T certainly has no eigen vectors. In this case the whole theory collapses into triviality. So we assume throughout this lesson that $H \neq [0]$.

Theorem 1: If T is a normal operator , x is a vector and u is a scalar , then $Tx = ux$ if and only if $T^*x = u\,x$. In particular

(1) x is a proper vector for T if and only if it is a proper vector for T*.

(2) u is a proper value of T if and only if it $\bar{u}$ is a proper value of T*.

Proof : By virtue of normality , $T^*T = TT^*$.

Since

$$(T - uI)^* = T^* - \bar{u}\,I^* = T^* - \bar{u}\,I.$$

we have

$$(T - uI)^*(T - uI) = (T^* - \bar{u}I)(T - uI)$$

$$= T^*T - uT^* - \bar{u}T + u\bar{u}I$$

and

$$(T - uI)(T - uI)^* = (T - uI)(T^* - \bar{u}I)$$

$$= TT^* - \bar{u}T - uT^* + u\bar{u}I$$

Since $TT^* = T^*T$, it follows that T - u I is normal. Hence

$$\|(T - uI)x\| = \|(T - uI)^*x\|$$

which in turn implies that Tx = u x if and only if T*x = $\bar{u}$ x. This proves (1) and (2).

Remark : Let H be a classical Hilbert space and x_1 , x_2 , ….. an orthonormal basis for H. Then one sided shift operator T defined by $Tx_k = x_{k+1}$ has no proper value.

Theorem 2: Let T be a normal operator on a Hilbert space H. Then there exists on orthonormal basis for H consisting of eigen vectors of T.

Proof : Let λ be an eigen value of T and suppose x is corresponding eigen vector. Thus we have $Tx = \lambda x$. Since x can not be zero , we can choose $x_1 = \frac{x}{\|x\|}$, If the dimension of H is 1 , then we are done. If not , we will proceed by induction. We shall assume that the theorem is true for all spaces of dimension less than H and then show that it follows for x from this assumption. Letting m = $[x_1] = [\alpha\ x_1 , \alpha \in F]$. The space spanned by x_1 , we have the following direct sum composition of H :

$$H = M \oplus M^{\perp} .$$

We must have then dim $M^{\perp}$ < dim H. Since x_1 is an eigen vector of T , we have $Tx_1 = \lambda\ x_1$ and therefore it is clear that M is invariant under T. But we know by theorem 1 that eigen vectors of T must also be eigen vectors for T*. Therefore M is invariant under T* also. Hence $M^{\perp}$ is invariant under T** = T. Thus we have

(i) M is invariant under T.

(ii) $M^{\perp}$ is invariant under T.

Thus we can say that M reduces T.

Consider now the restriction of T to $M^{\perp}$ denoted by $T/M^{\perp}$ where $T/M^{\perp} : M^{\perp} \rightarrow M^{\perp}$. Since T is normal , $T/M^{\perp}$ is also normal since $M^{\perp}$ reduces T. Now since dim $M^{\perp}$ < dim H , we can apply the induction hypothesis to assert the existence of an orthonormal basis for $M^{\perp}$ consisting of eigen vector for $T/M^{\perp}$; $\{x_1 , x_2 , \ldots , x_n\}$. Eigen vectors of $T/M^{\perp}$ however must also the eigen vector of T. Hence for the entire space , we have $(x_1 , x_2 , \ldots, x_n)$ as orthonormal basis of eigen vectors of T. Hence the result.

Spectral Theorem for Finite Dimensional spaces

Definition : The set of eigen values of an operator T is called its spectrum or point spectrum and is denoted by (T).

Statement of Spectral Theorem

Let λ_1 , λ_2 ,…. , λ_n be the eigen values of an operator T and let M_1 , M_2 , ….., M_n be their corresponding eigen spaces. If P_1 , P_2 ,….. , P_n are the projections on these eigen spaces , then the following three statements are equivalent to one another.

(1) M_i′s are pairwise orthogonal and span H.

(2) P_i′s are pairwise orthogonal , that is $P_i P_j = 0$ for $i \neq j$ and $I = P_1 + P_2 + \ldots\ldots + P_n$ and also

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots\ldots + \lambda_n P_n$$

(3) T is normal.

Proof : (1) $\Rightarrow$ (2) , BY (1) every vector x in H can be expressed uniquely in the form

$$x = x_1 + x_2 + \ldots\ldots + x_n \quad , \qquad (4)$$

where $x_i \in M_i$ for each i and x_i’s are pairwise orthogonal. Further (1) if $M_i \perp M_j$, $i \neq j$ then $M_j \subset M_i$. Then since $P_j x = M_j$ for every x , we have $P_i P_j x = 0$ for any x and $P_i P_j = 0$ for $i \neq j$. This proves that P_i’s are pairwise orthogonal. Applying P_i to both sides of (4) , we have

$$P_i x = P_i x_1 + P_i x_2 + \ldots\ldots + P_i x_n$$

$$= 0 + 0 + \ldots\ldots + P_i x_i + \ldots\ldots + 0$$

$$= x_i \qquad \text{for any } i.$$

Hence we can write any x as

$$x = P_1 x + P_2 x + \ldots\ldots + P_n x$$

or $\quad I x = P_1 x + P_2 x + \ldots\ldots + P_n x \quad$ for identity operator T.

or $\quad I x = (P_1 + P_2 + \ldots\ldots + P_n x) x$

Since this is true for any $x \in H$, we conclude that

$$I = P_1 + P_2 + \ldots\ldots + P_n .$$

Further applying T to x in (4) , we have

$$T x = T x_1 + T x_2 + \ldots\ldots + T x_n$$

$$= \lambda_1 x_1 + \lambda_2 x_2 + \ldots\ldots + \lambda_n x_n$$

$$= \lambda_1 P_1 x + \lambda_2 P_2 x + \ldots\ldots + \lambda_n P_n x$$

$$= (\lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots + \lambda_n P_n) x$$

for every x and so

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots\ldots + \lambda_n P_n \qquad (5)$$

The representation (5) for an operator T , when it exists is called the Spectral Representation or Spectral form of T.

(2) $\Rightarrow$ (3) , it follows from

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots + \lambda_n P_n$$

That

$$T^* = \bar{\lambda}_1 P_1^* + \bar{\lambda}_2 P_2^* + \ldots\ldots\ldots + \bar{\lambda}_n P_n^*$$

$$= \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \ldots\ldots + \bar{\lambda}_n P_n$$

Now since by (2) $P_i P_j = 0$ for $i \neq j$, we have

$$TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots + \lambda_n P_n)(\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \ldots\ldots + \bar{\lambda}_n P_n)$$

$$= |\lambda_1|^2 P_1^2 + \ldots\ldots\ldots + |\lambda_n|^2 P_n^2$$

$$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots\ldots\ldots + |\lambda_n|^2 P_n$$

and similarly

$$T^*T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots\ldots\ldots + |\lambda_n|^2 P_n$$

and therefore

$$TT^* = T^*T .$$

Proving that T is normal.

(3) $\Rightarrow$ (1) Suppose that T is normal.

We shall prove first that $M_i \perp M_j$ for $i \neq j$. Given $x_i \in M_i$, $x_j \in M_j$, it is sufficient to show that $x_i \perp x_j$. Since $x_i \in M_i$, $x_j \in M_j$, we have $Tx_i = \lambda_i x_i$, $Tx_j = \lambda_j x_j$. Since T is normal $Tx_j = \lambda_j x_j \Rightarrow T^*x_j = \bar{\lambda}_j x_j$ and so

$$(T x_i , x_j) = (x_i , T^* x_j)$$

or $(\lambda_i x_i , x_j) = (x_i , \bar{\lambda}_j x_j)$

or $\lambda_i (x_i , x_j) = \lambda_j (x_i , x_j)$

or $(\lambda_i - \lambda_j) (x_i , x_j) = 0$

Since $\lambda_i \neq \lambda_j$, it follows that $(x_i , x_j) = 0$ and hence $x_i \perp x_j$. This proves that $M_i \perp M_j$ for $i \neq j$ and so M_i's are pairwise orthogonal. It remains to prove that T is normal , then M_i's span H that is $H = M_1 + M_2 + \ldots\ldots + M_n$. We have just shown that M_i's are pairwise orthogonal. This implies that P_i's are pairwise orthogonal. Therefore $M = M_1 + M_2 + \ldots\ldots.. + M_n$ is a closed linear subspace of H and its associated projection is $P = P_1 + P_2 + \ldots\ldots\ldots + P_n$. Also we know that if T is normal , then M_i reduces T. Therefore $T P_i = P_i T$ for each P_i , it follows from this that $T P = P T$ and hence M reduces T and so by definition M is invariant under T. If $M \neq (0)$, then since all the eigen vectors of T are contained in M , the restriction of T to M is an operator (normal) on a non – trivial finite dimensional Hilbert space which has no eigen vectors and hence no eigen values. But this is a contradiction to the fact that there exists an orthonormal basis for H consisting of eigen vectors of normal operator T. Hence $M = (0)$ and so $M = H$ and hence $H = M_1 + M_2 + \ldots\ldots + M_n$ which shows that M_i's span H. Hence the result.